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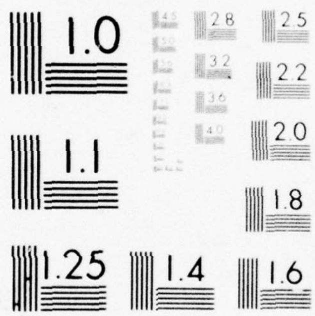

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SOME EXTENSIONS OF MARKOV REPAIR MODELS

by

YUKIO HATOYAMA

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Gerald J. Lieberman, Project Director

DEPARTMENT OF OPERATIONS RESEARCH
AND
DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

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CHAPTER 1

INTRODUCTION

In this paper two different types of discrete time Markov maintenance models having repair facilities available will be investigated. Because of their wide applicability in the practical world, a number of authors have studied optimization problems for discrete time machine maintenance models. Their main concern has been on the structure of an optimal policy, and a simple repair rule called a control limit policy has been introduced. When there is an operating machine, with its condition deteriorating as time goes on, a control limit policy is defined as a policy where a repair or replacement action is taken if and only if the degree of deterioration of the operating machine exceeds some critical value. Control limit policies are intuitively appealing policies, and sufficient conditions under which a control limit policy is optimal have been derived.

In 1963 Derman [2] introduced the basic maintenance model of this type. In his model, two actions are available at each time period, i.e. replace the current machine, or leave the current machine as is. If a replacement action is taken, a new machine replaces the old one, and the new machine begins to operate in its best condition at the beginning of the next period. If the nonreplacement action is chosen, the machine

keeps operating, and its degree of deterioration evolves from i to j in one period with transition probability p_{ij}, $0 \leq i, j \leq I$. He developed the notion of an increasing failure rate (IFR) distribution, which plays an important role in reliability theory (see Barlow [1]), for a Markov chain with a finite number of states. A Markov chain is said to be IFR if a higher state is associated with a greater chance of further deterioration. More precisely, a Markov chain with the transition matrix $\{p_{ij}\}$ is said to be IFR if $P_i(\cdot)$ is stochastically smaller than $P_{i+1}(\cdot)$ for $0 \leq i \leq I-1$, and can be written as

$$P_i(\cdot) \subset P_{i+1}(\cdot) , \qquad 0 \leq i \leq I-1,$$

where

$$P_i(k) = \sum_{j \leq k} p_{ij}, \qquad 0 \leq i \leq I .$$

Note that if $F(t)$ and $G(t)$ are distribution functions, $F(\cdot) \subset G(\cdot)$ if and only if $F(t) \geq G(t)$ for any t. Assuming a simple cost structure and the IFR property on the transition probabilities, Derman showed the optimality of a control limit policy.

Kolesar [9] extended the basic model by introducing state dependent operating costs, and Kalymon [6] further generalized the cost structure by allowing replacement costs to be stochastic. Klein [8] expanded Derman's model to include a costly inspection. Eppen [4], Taylor [14], and Rosenfield [12] developed their models along this line with emphasis on trying to find simple types of policies which optimize their models.

A joint replacement and stocking problem was introduced by Derman and Lieberman [3], which was generalized by Ross [13] to allow for deterioration of a machine.

The aforementioned models have the properties that the amount of time needed for the repair of a machine is one unit of time, and an unlimited supply of new spares is available. In that sense, they are replacement models rather than repair models. In 1973 Kao [7] introduced a semi-Markovian approach to the basic model. In his model, the repair time of a machine is no longer instantaneous but takes some random time according to its semi-Markovian nature, while the supply of new spares is kept unlimited. Hatoyama [5] investigated some machine repair models assuming that the repair time of a machine has a geometric distribution. A model with finite number of spare machines was also considered there.

In Chapter 2 of this paper, the basic machine repair problem is extended to allow for the existence of several repair shops. The repair time distributions depend on the type of repair work required on a machine rather than its condition. Here the condition of the machine regulates the type of repair work required in a probabilistic manner rather than deterministically. This extension seems suitable for the maintenance problems where the state of the machine such as its age does not completely coincide with the type of repair work on it. In a manner similar to that presented for the type of models treated in [5] control limit policy is defined, and sufficient conditions for the optimality of such control limit policies are obtained.

Chapter 3 considers the machine repair problem in the context of optimal control of queues. Since the number of machines in the system is fixed and finite, the models treated here necessarily become discrete time closed queueing systems. There appears to have been almost no research in optimization of discrete time closed queueing systems. Discrete time queueing systems have been investigated by Magazine [10], [11], but his models are open queueing systems. Models for the optimal control of closed queueing systems have been studied by Torbett [15], but his analysis is time continuous. In our models, the decision maker has an option of opening or closing the repair shop when there are machines waiting for repair service, as well as the option of repairing or leaving an operating machine alone. A set up cost for opening a closed gate, a shut down cost for closing an open gate, a holding cost for holding machines in the repair system, and a service cost for operating an open repair shop are introduced as is done in the usual queueing control models. A two-dimensional control limit policy is defined, and sufficient conditions for the existence of such a policy minimizing the total expected α-discounted cost are derived.

CHAPTER 2

MULTI-REPAIR-TYPE MAINTENANCE MODELS

This chapter treats maintenance models when there is more than one possible type of repair in the system. Here, the repair time distributions, material cost, and labor cost all depend on the type of repair work required on the machine. Sufficient conditions which result in the optimality of control limit policies of some kind are obtained. A generalized model is then considered. Finally, a model where there is one repairman for each type of repair shop is studied.

2.1. The Model

We consider a basic multi-repair-type Markov machine maintenance model. The flow of machines in the system is schematically shown in Fig. 2.1.

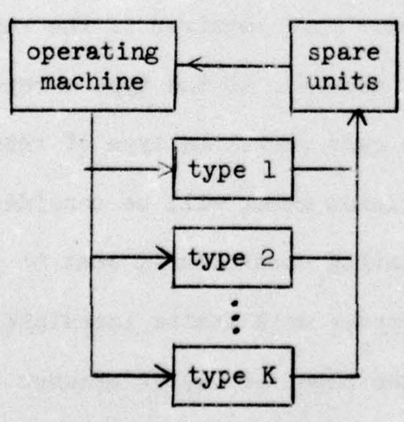

Figure 2.1. A Multi-Repair-Type Machine Maintenance System.

There is an operating machine and S $(S \geq 1)$ spare machines in the system. The system is observed periodically and at the beginning of each period an operating machine is classified as being in one of $I+1$ $(I \geq 1)$ states, with each state showing the degree of deterioration. 0 represents a machine

in its best condition, while I denotes its failure. When a machine is operating, two choices are available at the beginning of each period: to let it keep operating, or to repair it. If the former decision is chosen, the state of the operating machine evolves from i to j in one unit of time according to the transition probability $p_{ij} \geq 0$. If the latter is selected, the machine is immediately sent to one of K $(K \geq 2)$ repair shops, and is replaced by one of the spare machines, if any is available. The new operating machine begins to operate in its best condition. Type 1 through type K repair shops are arranged so that type 1 represents the easiest repair work and type K the hardest. If a machine in the i-th operating condition is chosen to be repaired, a type k repair facility is required with probability $p_k^{(i)} \geq 0$. Here, we assume $\sum_{k=1}^{K} p_k^{(i)} = 1$ for $0 \leq i \leq I$. A failed machine must be repaired. Furthermore, suppose the type of repair required if a repair decision is made is known to the decision maker before he takes an action. Then the total information available to the decision maker at the beginning of each period is the condition of an operating machine, the type of repair work required if the repair decision is chosen, and the number of machines in the type k repair shop for each $1 \leq k \leq K$. Later, the case where the type of repair work required is not known to the decision maker will be considered.

If a machine in the i-th operating condition is sent to the type k $(1 \leq k \leq K)$ repair shop, the repair work starts immediately, and its repair time T_k, which is independent of i, is assumed to be a random variable having a geometric distribution with parameter q_k, i.e.,

$$P\{T_k = j\} = q_k(1-q_k)^{j-1}, \qquad j = 1,2,\ldots .$$

If all the machines are in the repair shops, the system fails since no machine is available. In that case we must wait until one of the machines is completely repaired. A penalty cost, P, is assessed per period during the system's failure. $A(i)$ is the operating cost for a machine in the i-th operating condition, $C(i,k)$ is the material cost for repairing a machine in the i-th operating condition in the type k repair shop, and $B(k)$ is the labor cost for the type k repair work for a single machine per period. The objective is to find a repair policy which minimizes the total α-discounted cost.

Before proceeding further it is useful to give an example in which the above model can be applied. Consider the problem of a taxicab driver who owns several cars. He continues to use one of his cars until he decides to repair it. He then begins to use another one, if any is available. We assume that the maintenance, or operating cost (excluding repair costs), of a car depends mainly on its age or the number of days after its last repair. However, the actual deterioration of the car does not necessarily coincide with its age. Among cars having the same age, some may require little repair work, while others may require substantial amounts of repair work. A car with i-th age requires type k repair work with probability $p_k^{(i)}$. If the taxicab driver is familiar with the repair job, he can get the information on the type of repair work required as well as the age of the car before he makes the decision on whether or not the car should be repaired. Otherwise he can utilize just the age of the car when he makes a decision.

The repair time distribution is assumed to be geometric, and its parameter depends on the type of repair work. Also, repair costs mainly depend on the type of repair work. If all the cars are in the repair facilities, no car is available to him, and hence, he loses some amount of the expected revenue. If he is interested in the total α-discounted cost, then finding the best repair schedule is an example of the type of problem which will be treated in this chapter.

The state of this system is represented by the K+2 vector

$$X_t = (X_t^0, X_t^1, X_t^2, \ldots, X_t^{K+1}) = (i, k, s_1, \ldots, s_K) ,$$

where at the beginning of the t-th period there is an operating machine whose operating condition is i, type k repair work is required if the repair decision is chosen, and s_j $(1 \leq j \leq K)$ machines are in the type j repair facility. Here

$$0 \leq i \leq I, \quad 1 \leq k \leq K, \quad 0 \leq \sum_{j=1}^{K} s_j \leq S, \quad s_j \geq 0 \quad (1 \leq j \leq K) .$$

For notational convenience, we define

$$X_t = (0, 1, s_1, \ldots, s_K), \qquad \sum_{j=1}^{K} s_j = S+1, \quad s_j \geq 0 \quad (1 \leq j \leq K)$$

if all the machines are in the repair facilities and none of the machines is operating at the beginning of the t-th period.

In order to simplify the repetitive use of the following sets, define

$$\mathscr{I} = \{i \mid 0 \le i \le I, \ i : \text{integer}\}$$

$$\mathscr{K} = \{k \mid 1 \le k \le K, \ k : \text{integer}\}$$

$$\mathscr{S}^m = \{(s_1, \ldots, s_K) \mid \sum_{k=1}^{K} s_k \le m, \ s_j \ (1 \le j \le K) : \text{nonnegative integer}\}$$

$$\mathscr{S}_0^m = \{(0, 1, s_1, \ldots, s_K) \mid \sum_{k=1}^{K} s_k = m, \ s_j \ (1 \le j \le K) : \text{nonnegative integer}\}.$$

Let $q_{ss'}^{(j)}$ be the probability that s' $(s' \le s)$ machines are still in the type j repair shop at the end of the period, given s machines are in the type j $(1 \le j \le K)$ repair shop at the beginning of a period. Then

$$q_{s_j s_j'}^{(j)} = \binom{s_j}{s_j'} (1 - q_j)^{s_j'} q_j^{s_j - s_j'} , \qquad 1 \le j \le K ,$$

since the repair time, T_j, has a geometric distribution with parameter q_j.

Before formulating the problem as a dynamic programming problem, we make the following simplification. When a new machine replaces the previously operating machine, the new one starts operating in its best condition at the beginning of the next period, and the least repair work, i.e., type 1 repair is required if it is instantaneously determined to be repaired.

Let $V_\alpha(i,k,s_1,\ldots,s_K;n)$ be the minimum expected n period α-discounted cost starting from state $(i,k,s_1,\ldots,s_K)$. Then by setting $V_\alpha(i,k,s_1,\ldots,s_K;0) = 0$ for any feasible $(i,k,s_1,\ldots,s_K)$, $V_\alpha(i,k,s_1,\ldots,s_K;n)$ $(n \ge 1)$ satisfies the following set of recursive equations:

$$V_\alpha(i,k,s_1,\dots,s_K;n)$$

$$= \min\{A(i) + \sum_{j=1}^{K} B(j)s_j + \alpha \sum_{i'=0}^{I} \sum_{k'=1}^{K} \sum_{s_1'=0}^{s_1} \cdots \sum_{s_K'=0}^{s_K} p_{ii'}$$

$$\cdot\, p_{k'}^{(i')} q_{s_1 s_1'}^{(1)} \cdots q_{s_K s_K'}^{(K)} V_\alpha(i',k',s_1',\dots,s_K';n-1),$$

$$C(i,k) + B(k) + \sum_{j=1}^{K} B(j)s_j + \alpha \sum_{s_1'=0}^{s_1} \cdots \sum_{s_k'=0}^{s_k+1} \cdots \sum_{s_K'=0}^{s_K} q_{s_1 s_1'}^{(1)}$$

$$\cdots q_{s_k+1,s_k'}^{(k)} \cdots q_{s_K s_K'}^{(K)} V_\alpha(0,1,s_1',\dots,s_K';n-1)\}$$

$$\text{for } (i,k,s_1,\dots,s_K) \in \mathcal{I} \times \mathcal{K} \times \mathcal{S}^S \qquad (2.1)$$

$$= P + \sum_{j=1}^{K} B(j)s_j + \alpha \sum_{s_1'=0}^{s_1} \cdots \sum_{s_K'=0}^{s_K} q_{s_1 s_1'}^{(1)} \cdots q_{s_K s_K'}^{(K)} V_\alpha(0,1,s_1',\dots,s_K';n-1)$$

$$\text{for } (i,k,s_1,\dots,s_K) \in \mathcal{S}_0^{S+1} .$$

By letting

$$R_\alpha(i,k,s_1,\dots,s_K;n) = \sum_{s_1'=0}^{s_1} \cdots \sum_{s_K'=0}^{s_K} q_{s_1 s_1'}^{(1)} \cdots q_{s_K s_K'}^{(K)} V_\alpha(i,k,s_1',\dots,s_K';n),$$

we can simplify (2.1) as below: For $n \geq 1$,

$V_\alpha(i,k,s_1,\ldots,s_K;n)$

$$= \min\{A(i) + \sum_{j=1}^{K} B(j)s_j + \alpha \sum_{i'=0}^{I} \sum_{k'=1}^{K} p_{ii'}p_{k'}^{(i')} R_\alpha(i',k',s_1,\ldots,s_K;n-1),$$

$$C(i,k) + B(k) + \sum_{j=1}^{K} B(j)s_j + \alpha R_\alpha(0,1,s_1,\ldots,s_k+1,\ldots,s_K;n-1)\}$$

$$\text{for } (i,k,s_1,\ldots,s_K) \in \mathcal{I} \times \mathcal{K} \times \mathcal{S}^S \qquad (2.2)$$

$$= P + \sum_{j=1}^{K} B(j)s_j + \alpha R_\alpha(0,1,s_1,\ldots,s_K;n-1)$$

$$\text{for } (i,k,s_1,\ldots,s_K) \in \mathcal{S}_0^{S+1} .$$

Let $V_\alpha(i,k,s_1,\ldots,s_K)$ be the total expected α-discounted cost starting from state $(i,k,s_1,\ldots,s_K) \in \mathcal{I} \times \mathcal{K} \times \mathcal{S}^S \cup \mathcal{S}_0^{S+1}$. Then

$$V_\alpha(i,k,s_1,\ldots,s_K) = \lim_{n\to\infty} V_\alpha(i,k,s_1,\ldots,s_K;n) .$$

Furthermore, the existence of a stationary policy minimizing the total expected α-discounted cost is guaranteed since this model is a Markov decision process.

Two kinds of control limit policies are now introduced.

Definition. An i control limit policy is a nonrandomized policy where there is a special operating condition i for each k $(1 \le k \le K)$, for each feasible $s = (s_1,\ldots,s_K)$, and for each period n $(n \ge 1)$, say $i_{k,s,n}$, such that for all $(i,k,s_1,\ldots,s_K)$ with $i < i_{k,s,n}$, the decision

at period n is to keep an operating machine in operation, and for all $(i,k,s_1,\ldots,s_K)$ with $i \geq i_{k,s,n}$ the decision at n is to repair it. A k control limit policy is a nonrandomized policy where there is a special type k of repair shop for each i $(0 \leq i \leq I)$, for each $s = (s_1,\ldots,s_K)$, and for each period n, say $k_{i,s,n}$, such that for all $(i,k,s_1,\ldots,s_K)$ with $k \leq k_{i,s,n}$, the decision at n is to repair an operating machine, and for all $(i,k,s_1,\ldots,s_K)$ with $k > k_{i,s,n}$ the decision at n is to keep it operating.

Sufficient conditions for the existence of a stationary i control limit policy which minimizes the total expected α-discounted cost of this model are obtained first through several lemmas shown next.

Lemma 2.1. Assume the following conditions hold:

1. $B(k)$ is nonnegative and nondecreasing in k for $1 \leq k \leq K$.
2. $q_k \geq q_{k+1}$ for $1 \leq k \leq K-1$.
3. $P \geq C(0,1)$.

Then for $(i,k',s_1,\ldots,s_K) \in \mathcal{I} \times \mathcal{K} \times \mathcal{S}^{S-1} \cup \mathcal{S}_0^S$, $n \geq 1$,

(a) $V_\alpha(i,k',s_1,\ldots,s_{k+1}+1,\ldots,s_K;n) \geq V_\alpha(i,k',s_1,\ldots,s_k+1,\ldots,s_K;n)$.

(b) $V_\alpha(i,k',s_1,\ldots,s_k+1,\ldots,s_K;n) \geq V_\alpha(i,k',s_1,\ldots,s_k,\ldots,s_K;n)$.

Proof. Proof is by mathematical induction. For $n = 1$, it is easy to check that both (a) and (b) hold for $(i,k',s_1,\ldots,s_K) \in \mathcal{I} \times \mathcal{K} \times \mathcal{S}^{S-1} \cup \mathcal{S}_0^S$, and the proof is omitted. Suppose both (a) and (b) hold for $n = m-1 \geq 1$. Consider the case for $n = m$. Take (a) first.

Notice that for $(i,k',s_1,\ldots,s_K) \in \mathcal{I} \times \mathcal{K} \times \mathcal{S}^{S-1} \cup \mathcal{S}_0^S$,

$$R_\alpha(i,k',s_1,\ldots,s_{k+1}+1,\ldots,s_K;m-1) - R_\alpha(i,k',s_1,\ldots,s_k+1,\ldots,s_K;m-1)$$

$$= (1-q_{k+1}) \sum_{s_1'=0}^{s_1} \cdots \sum_{s_K'=0}^{s_K} q_{s_1 s_1'}^{(1)} \cdots q_{s_K s_K'}^{(K)}$$

$$\cdot (V_\alpha(i,k',s_1',\ldots,s_{k+1}'+1,\ldots,s_K';m-1) - V_\alpha(i,k',s_1',\ldots,s_K';m-1))$$

$$- (1-q_k) \sum_{s_1'=0}^{s_1} \cdots \sum_{s_K'=0}^{s_K} q_{s_1 s_1'}^{(1)} \cdots q_{s_K s_K'}^{(K)}$$

$$\cdot (V_\alpha(i,k',s_1',\ldots,s_k'+1,\ldots,s_K';m-1) - V_\alpha(i,k',s_1',\ldots,s_K';m-1)),$$

by Lemma 3.3 of [5]

$$\geq (1-q_{k+1}) \sum_{s_1'=0}^{s_1} \cdots \sum_{s_K'=0}^{s_K} q_{s_1 s_1'}^{(1)} \cdots q_{s_K s_K'}^{(K)}$$

$$\cdot (V_\alpha(i,k',s_1',\ldots,s_{k+1}'+1,\ldots,s_K';m-1) - V_\alpha(i,k',s_1',\ldots,s_k'+1,\ldots,s_K';m-1)),$$

from 2 and by inductive assumption on (b) for $n = m-1$

≥ 0 , by inductive assumption on (a) for $n = m-1$.

Using the above result, for $(i,k',s_1,\ldots,s_K) \in \mathcal{S}_0^S$,

$$V_\alpha(0,1,s_1,\ldots,s_{k+1}+1,\ldots,s_K;m) - V_\alpha(0,1,s_1,\ldots,s_k+1,\ldots,s_K;m)$$

$$= P + \sum_{j=1}^{K} B(j)s_j + B(k+1) + \alpha R_\alpha(0,1,s_1,\ldots,s_{k+1}+1,\ldots,s_K;m-1)$$

$$- (P + \sum_{j=1}^{K} B(j)s_j + B(k) + \alpha R_\alpha(0,1,s_1,\ldots,s_k+1,\ldots,s_K;m-1))$$

$$= B(k+1) - B(k) + \alpha(R_\alpha(0,1,s_1,\ldots,s_{k+1}+1,\ldots,s_K;m-1)$$
$$- R_\alpha(0,1,s_1,\ldots,s_k+1,\ldots,s_K;m-1))$$
≥ 0, from 1.

For $(i,k',s_1,\ldots,s_K) \in \mathcal{I}\times\mathcal{K}\times\mathcal{S}^{S-1}$, we compare the corresponding values term by term (see [5]).

$$\langle V_\alpha(i,k',s_1,\ldots,s_{k+1}+1,\ldots,s_K;m)\rangle_{1\text{-st}} - \langle V_\alpha(i,k',s_1,\ldots,s_k+1,\ldots,s_K;m)\rangle_{1\text{-st}}$$

$$= A(i) + \sum_{j=1}^{K} B(j)s_j + B(k+1) + \alpha \sum_{i'=0}^{I}\sum_{j=1}^{K} p_{ii'}p_j^{(i')}$$

$$\cdot R_\alpha(i',j,s_1,\ldots s_{k+1}+1,\ldots,s_K;m-1) - (A(i) + \sum_{j=1}^{K} B(j)s_j + B(k)$$

$$+ \alpha \sum_{i'=0}^{I}\sum_{j=1}^{K} p_{ii'}p_j^{(i')} R_\alpha(i',j,s_1,\ldots,s_k+1,\ldots,s_K;m-1))$$

$$= B(k+1) - B(k) + \alpha \sum_{i'=0}^{I}\sum_{j=1}^{K} p_{ii'}p_j^{(i')}(R_\alpha(i',j,s_1,\ldots,s_{k+1}+1,\ldots,s_K;m-1)$$
$$- R_\alpha(i',j,s_1,\ldots,s_k+1,\ldots,s_K;m-1))$$

≥ 0 .

In the similar manner, we have

$$\langle V_\alpha(i,k',s_1,\ldots,s_{k+1}+1,\ldots,s_K;m)\rangle_{2\text{-nd}} - \langle V_\alpha(i,k',s_1,\ldots,s_k+1,\ldots,s_K;m)\rangle_{2\text{-nd}}$$

≥ 0.

Hence also for $(i,k',s_1,\ldots,s_K) \in \mathcal{I}\times\mathcal{K}\times\mathcal{S}^{S-1}$,

$$V_\alpha(i,k',s_1,\ldots,s_{k+1}+1,\ldots,s_K;m) \geq V_\alpha(i,k',s_1,\ldots,s_k+1,\ldots,s_K;m)$$

which shows that (a) holds for $n = m$.

Consider the (b) case for $n = m$. For $(i,k',s_1,\ldots,s_K) \in \mathcal{I}\times\mathcal{K}\times\mathcal{S}^{S-1}$, we again compare the corresponding values term by term.

$$\langle V_\alpha(i,k',s_1,\dots,s_k+1,\dots,s_K;m)\rangle_{1\text{-st}} - \langle V_\alpha(i,k',s_1,\dots,s_k,\dots,s_K;m)\rangle_{1\text{-st}}$$

$$= B(k) + \alpha \sum_{i'=0}^{I} \sum_{j=1}^{K} p_{ii'} p_j^{(i')} (R_\alpha(i',j,s_1,\dots,s_k+1,\dots,s_K;m-1)$$

$$- R_\alpha(i',j,s_1,\dots,s_k,\dots,s_K;m-1))$$

≥ 0, from 1 and inductive assumption on (b) for $n = m-1$.

Similarly, the difference of the corresponding second terms can be shown to be nonnegative.

Lastly, for $(i,k',s_1,\dots,s_K) \in \mathcal{S}_0^S$,

$$V_\alpha(0,1,s_1,\dots,s_k+1,\dots,s_K;m) - V_\alpha(0,1,s_1,\dots,s_k,\dots,s_K;m)$$

$$= P + \sum_{j=1}^{K} B(j)s_j + B(k) + \alpha R_\alpha(0,1,s_1,\dots,s_k+1,\dots,s_K;m-1)$$

$$- \min\{A(0) + \sum_{j=1}^{K} B(j)s_j + \alpha \sum_{i'=0}^{I} \sum_{j=1}^{K} p_{0i'} p_j^{(i')} R_\alpha(i',j,s_1,\dots,s_K;m-1),$$

$$C(0,1) + B(1) + \sum_{j=1}^{K} B(j)s_j + \alpha R_\alpha(0,1,s_1+1,s_2,\dots,s_K;m-1)\}$$

$$\geq P + \sum_{j=1}^{K} B(j)s_j + B(k) + \alpha R_\alpha(0,1,s_1,\dots,s_k+1,\dots,s_K;m-1)$$

$$- (C(0,1) + B(1) + \sum_{j=1}^{K} B(j)s_j + \alpha R_\alpha(0,1,s_1+1,s_2,\dots,s_K;m-1))$$

$\geq (P - C(0,1)) + (B(k) - B(1))$ by inductive assumption on (a) for $n = m-1$

≥ 0, from 1 and 3.

Hence, for $(i,k',s_1,\dots,s_K) \in \mathcal{I} \times \mathcal{K} \times \mathcal{S}^{S-1} \cup \mathcal{S}_0^S$, (b) holds for $n = m$, *completing the mathematical induction and yielding that both assertions* (a) and (b) hold for $n \geq 1$. □

Lemma 2.2. Suppose conditions 1, 2 and 3 of Lemma 2.1 hold. Furthermore, assume

4. $C(i,k)$ is nondecreasing in k $(1 \le k \le K)$ for each fixed i $(0 \le i \le I)$.

Then $V_\alpha(i,k,s_1,\dots,s_K;n)$ is nondecreasing in k $(1 \le k \le K)$ for each fixed $(i,s_1,\dots,s_K) \in \mathcal{I} \times \mathcal{S}^S$ and for $n \ge 1$.

Proof. Mathematical induction is applied. For $n = 1$, for $(i,k,s_1,\dots,s_K) \in \mathcal{I} \times \mathcal{K} \times \mathcal{S}^S$,

$$V_\alpha(i,k,s_1,\dots,s_K;1) = \min\{A(i) + \sum_{j=1}^{K} B(j)s_j, C(i,k) + B(k) + \sum_{j=1}^{K} B(j)s_j\}$$

is clearly nondecreasing in k, from 1 and 4. Suppose the assertion holds for $n = m-1 \ge 1$, and consider the case for $n = m$. The first term of the right hand side of $V_\alpha(i,k,s_1,\dots,s_K;m)$ for $(i,k,s_1,\dots,s_K) \in \mathcal{I} \times \mathcal{K} \times \mathcal{S}^S$ is independent of k, so it is enough to check the second term. For $(i,s_1,\dots,s_K) \in \mathcal{I} \times \mathcal{S}^S$, and for $1 \le k \le K-1$,

$$\langle V_\alpha(i,k+1,s_1,\dots,s_K,m)\rangle_{2\text{-nd}} - \langle V_\alpha(i,k,s_1,\dots,s_K;m)\rangle_{2\text{-nd}}$$

$$= C(i,k+1) + B(k+1) + \sum_{j=1}^{K} B(j)s_j + \alpha R_\alpha(0,1,s_1,\dots,s_{k+1}+1,\dots,s_K;m-1)$$

$$- (C(i,k) + B(k) + \sum_{j=1}^{K} B(j)s_j + \alpha R_\alpha(0,1,s_1,\dots,s_k+1,\dots,s_K;m-1))$$

≥ 0, from 1, 4 and the result (a) of Lemma 2.1.

Hence the assertion holds for $n = m$, completing the mathematical induction and the proof of the lemma. □

Lemma 2.3. Suppose a function $V(i,k)$ is nondecreasing in i $(0 \leq i \leq I)$ for each fixed k $(1 \leq k \leq K)$ and nondecreasing in k $(1 \leq k \leq K)$ for each fixed i $(0 \leq i \leq I)$, and suppose

5. $P_i(\cdot) \subset P_{i+1}(\cdot)$ for $0 \leq i \leq I-1$ where $P_i(j) = \sum_{i' \leq j} p_{ii'}$, $0 \leq j \leq I$.

6. $P^{(i)}(\cdot) \subset P^{(i+1)}(\cdot)$ for $0 \leq i \leq I-1$, where $P^{(i)}(k) = \sum_{k' \leq k} p_{k'}^{(i)}$, $1 \leq k \leq K$.

Then, $\sum_{i'=0}^{I} \sum_{k'=1}^{K} p_{ii'}\, p_{k'}^{(i')} V(i',k')$ is nondecreasing in i for $0 \leq i \leq I$.

Proof. For $0 \leq i' \leq I-1$,

$$\sum_{k'=1}^{K} p_{k'}^{(i'+1)} V(i'+1,k') \geq \sum_{k'=1}^{K} p_{k'}^{(i'+1)} V(i',k'),$$

since $V(i',k')$ is nondecreasing in i' for fixed k'

$$\geq \sum_{k'=1}^{K} p_{k'}^{(i')} V(i',k') ,$$

by 6 and since $V(i',k')$ is nondecreasing in k'

which gives that $g(i') = \sum_{k'=1}^{K} p_{k'}^{(i')} V(i',k')$ is nondecreasing in i' $(0 \leq i' \leq I)$. From 5, this implies that $\sum_{i'=0}^{I} p_{ii'}\, g(i')$ is nondecreasing in i $(0 \leq i \leq I)$, which is what we want. □

Lemma 2.4. Assume conditions 1 though 6 of the last three lemmas hold, and furthermore suppose

7. $C(i,k)$ is nondecreasing in i $(0 \leq i \leq I)$ for each fixed k $(1 \leq k \leq K)$.

8. $A(i)$ is nondecreasing in i for $0 \leq i \leq I$.

Then $V_\alpha(i,k,s_1,\ldots,s_K;n)$ is nondecreasing in i $(0 \leq i \leq I)$ for each fixed $(k,s_1,\ldots,s_K) \in \mathcal{K} \times \mathcal{S}^S$ and for $n \geq 1$.

Proof. Mathematical induction is again used. For $n = 1$, for $(k,s_1,\ldots,s_K) \in \mathcal{K} \times \mathcal{S}^S$ it is obvious that $V_\alpha(i,k,s_1,\ldots,s_K;1)$ is nondecreasing in i from 7 and 8. Suppose the assertion is true for $n = m-1 \geq 1$, and consider the case for $n = m$. Then for $(k,s_1,\ldots,s_K) \in \mathcal{K} \times \mathcal{S}^S$,

$$V_\alpha(i,k,s_1,\ldots,s_K;m)$$

$$= \min\{A(i) + \sum_{j=1}^{K} B(j)s_j + \alpha \sum_{i'=0}^{I} \sum_{k'=1}^{K} p_{ii'}p_{k'}^{(i')} R_\alpha(i',k',s_1,\ldots,s_K;m-1),$$

$$C(i,k) + B(k) + \sum_{j=1}^{K} B(j)s_j + \alpha R_\alpha(0,1,s_1,\ldots,s_k+1,\ldots,s_K;m-1)\} .$$

The second term is obviously nondecreasing in i from 7, and as $V_\alpha(i,k,s_1,\ldots,s_K;m-1)$ is nondecreasing in i by the inductive assumption, and also is nondecreasing in k by Lemma 2.2, so is $R_\alpha(i,k,s_1,\ldots,s_K;m-1)$ by definition. Using Lemma 2.3 indicates that $\sum_{i'=0}^{I} \sum_{k'=1}^{K} p_{ii'}p_{k'}^{(i')} R_\alpha(i',k',s_1,\ldots,s_K;m-1)$ is nondecreasing in i $(0 \leq i \leq I)$. As $A(i)$ is nondecreasing in i $(0 \leq i \leq I)$ from 8, so is the first term of $V_\alpha(i,k,s_1,\ldots,s_K;m)$, yielding that $V_\alpha(i,k,s_1,\ldots,s_K;m)$ is nondecreasing in i for $0 \leq i \leq I$, which completes the mathematical induction and the proof. □

Now using the above lemmas, we can prove the following main theorem.

Theorem 2.5. Assume the following conditions hold:

1. $B(k)$ is nonnegative and nondecreasing in k for $1 \leq k \leq K$.
2. $C(i,k)$ is nondecreasing in k $(1 \leq k \leq K)$ for each fixed i $(0 \leq i \leq I)$.
3. $C(i,k)$ is nondecreasing in i $(0 \leq i \leq I)$ for each fixed k $(1 \leq k \leq K)$.
4. $A(i) - C(i,k)$ is nondecreasing in i $(0 \leq i \leq I)$ for each fixed k $(1 \leq k \leq K)$.
5. $P \geq C(0,1)$.
6. $q_k \geq q_{k+1} > 0$ for $1 \leq k \leq K-1$.
7. $P_i(\cdot) \subset P_{i+1}(\cdot)$ for $0 \leq i \leq I-1$.
8. $P^{(i)}(\cdot) \subset P^{(i+1)}(\cdot)$ for $0 \leq i \leq I-1$.

Then there exists a stationary i control limit policy which minimizes the total expected α-discounted cost of the basic multi-repair-type maintenance model.

Proof. First notice that if all the conditions are satisfied, Lemma 2.4 holds since conditions 3 and 4 of this theorem imply condition 8 of Lemma 2.4 and all other conditions are exactly the same.

For $(i,k,s_1,\ldots,s_K) \in \mathcal{I} \times \mathcal{K} \times \mathcal{S}^S$ and for $n \geq 0$, let

$$f_{n+1}(i,k,s_1,\ldots,s_K)$$
$$= \langle V_\alpha(i,k,s_1,\ldots,s_K;n+1)\rangle_{1\text{-st}} - \langle V_\alpha(i,k,s_1,\ldots,s_K;n+1)\rangle_{2\text{-nd}}$$

$$= A(i) + \sum_{j=1}^{K} B(j)s_j + \alpha \sum_{i'=0}^{I} \sum_{k'=1}^{K} p_{ii'}p_{k'}^{(i')} R_\alpha(i',k',s_1,\ldots,s_K,n)$$

$$- (C(i,k) + B(k) + \sum_{j=1}^{K} B(j)s_j + \alpha R_\alpha(0,1,s_1,\ldots,s_k+1,\ldots,s_K;n)).$$

Now by Lemmas 2.2 and 2.4, $V_\alpha(i',k',s_1,\ldots,s_K;n)$ $(n \geq 0)$ is both nondecreasing in i' $(0 \leq i' \leq I)$ for fixed k', and nondecreasing in k' $(1 \leq k' \leq K)$ for fixed i'. So is $R_\alpha(i',k',s_1,\ldots,s_K;n)$ $(n \geq 0)$, by definition. Therefore, using Lemma 2.3, we have that $\sum_{i'=0}^{I} \sum_{k'=1}^{K} p_{ii'}p_{k'}^{(i')} R_\alpha(i',k',s_1,\ldots,s_K;n)$ $(n \geq 0)$ is nondecreasing in i $(0 \leq i \leq I)$. The only other expression containing i is $A(i) - C(i,k)$, which is assumed to be nondecreasing in i $(0 \leq i \leq I)$, by 4. Hence for $n \geq 1$, $f_n(i,k,s_1,\ldots,s_K)$ is nondecreasing in i $(0 \leq i \leq I)$ for each fixed $(k,s) = (k,s_1,\ldots,s_K) \in \mathcal{K} \times \mathcal{S}^S$. That means, at the beginning of each n $(n \geq 1)$ period problem, for each $(k,s) \in \mathcal{K} \times \mathcal{S}^S$, there exists an $i_{k,s,n}$ such that $\langle V_\alpha(i,k,s_1,\ldots,s_K;n)\rangle_{2\text{-nd}}$ is smaller than or equal to $\langle V_\alpha(i,k,s_1,\ldots,s_K;n)\rangle_{1\text{-st}}$, i.e., to repair an operating machine is optimal if and only if $i \geq i_{k,s,n}$. Thus, the existence of an i control limit policy optimizing a finite horizon problem is guaranteed. Now using the fundamental results of Markov decision theory, the existence of a stationary i control limit policy minimizing the infinite horizon problem can be easily obtained (see Theorem 2.2 of [5]). □

Interpretation of each condition is given now. Conditions 1, 2 and 6 characterize the fact that the types of repair works are arranged so that type 1 is the easiest and type K is the hardest, since they say

that the labor cost, the material cost, and the expected length of repair time, all increases as the type number of repair work increases. Condition 3 states that the worse the machine is, the more expensive its material cost is. Condition 4 requires that the operating cost must increase more than the increase in material cost for each type of repair. Condition 5 gives a simple lower bound on the penalty cost. Condition 7 is the so-called IFR property of a deteriorating system. Lastly condition 8 indicates that the worse the state of the machine, the harder it is to repair (stochastically). All of them seem reasonable, and none of them is seriously restrictive.

Suppose all the conditions of Theorem 2.5 hold. Then for each $(k,s) \in \mathcal{K} \times \mathcal{S}^S$, there exists an $i_{k,s}$ such that for all (i,k,s) with $i < i_{k,s}$ to keep a machine in operation is optimal, and for all (i,k,s) with $i \geq i_{k,s}$ to repair it is optimal at any period. If, in addition, we can show that the optimal policy obtained is also a k control limit policy, then $i_{k,s}$ becomes nondecreasing in k $(1 \leq k \leq K)$ for each fixed $s \in \mathcal{S}^S$. Then one realization of an optimal policy for each fixed $s \in \mathcal{S}^S$ is as shown in Fig. 2.2.

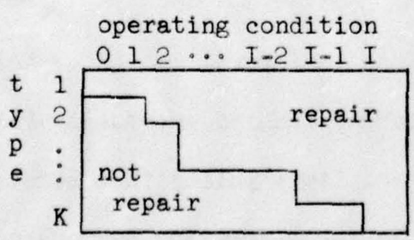

Figure 2.2. An Example of an Optimal Policy.

The above argument is justified by the following theorem.

Theorem 2.6. If conditions 1, 2, 5 and 6 of Theorem 2.5 hold, then there exists a stationary k control limit policy minimizing the total expected α-discounted cost of the multi-repair-type maintenance model.

Proof. Just as in the previous theorem, it is enough to check that $f_n(i,k,s_1,\ldots,s_K)$ is nonincreasing in k $(1 \le k \le K)$ for each $(i,s_1,\ldots,s_K) \in \mathcal{I} \times \mathcal{S}^S$ and $n \ge 1$. Now if conditions 1, 5 and 6 of Theorem 2.5 hold, then Lemma 2.1 holds and hence, $R_\alpha(0,1,s_1,\ldots,s_k+1,\ldots,s_K;n)$ $(n \ge 1)$ is nondecreasing in k $(1 \le k \le K)$. From conditions 1 and 2 of Theorem 2.5, both $B(k)$ and $C(i,k)$ are nondecreasing in k $(1 \le k \le K)$. As all other terms do not contain k in their expressions, we have $f_n(i,k,s_1,\ldots,s_K)$ is nonincreasing in k $(1 \le k \le K)$. □

The long-run expected average cost version of Theorems 2.5 and 2.6 can be easily obtained and stated as below:

Theorem 2.7. Suppose all the conditions in Theorem 2.5 (2.6) hold, and furthermore suppose that any operating machine eventually fails. Then there exists a stationary i control limit policy (k control limit policy, respectively) minimizing the long-run expected average cost of the basic multi-repair-type maintenance model.

Proof. Since any operating machine eventually fails, and a failed machine must be repaired at once, and $q_k > 0$ for any k $(1 \le k \le K)$, $(i,k,s_1,\ldots,s_K) = (0,1,0,\ldots,0)$ is accessible from every other state

no matter what stationary policy is employed. Hence, the existence of a stationary policy optimizing the problem is guaranteed. The rest of the proof is just the same as that of Theorem 2.3 of [5] and hence can be omitted. □

In the aforementioned model, we assumed that the type of repair work required was known to the decision maker before he made a decision. Now we briefly discuss the case where the type of repair work required is not known to the decision maker. In this modified model, if an operating machine in the i-th operating condition is chosen to be repaired, it is randomly sent to the type k repair shop with probability $p_k^{(i)}$, and the decision maker has no knowledge of where it is to be sent before he makes a decision. We can define an i control limit policy for this model as in the previous model, and the sufficient conditions under which an optimal policy is of an i control limit form are to be studied. Though the model description is not much different from the basic model, the analysis becomes much more complicated, and some bounding technique on the value of the cost criterion must be applied to derive sufficient conditions (see [5]). The main difference in the results is in the condition which relates the operating cost and other repair costs. The following inequality replaces condition 4 of Theorem 2.5:

$$\begin{aligned} A(i+1) - A(i) \geq & \sum_{k=1}^{K} p_k^{(i+1)}(C(i+1,k) + B(k)) - \sum_{k=1}^{K} p_k^{(i)}(C(i,k) + B(k)) \\ & + \sum_{k=1}^{K} \max(p_k^{(i+1)} - p_k^{(i)}, 0)\ (\frac{1}{q_k} - 1)(P + B(k) \\ & - \min\{A(0),\ \sum_{k=1}^{K} p_k^{(0)}\ C(0,k) + B(1)\}) , \quad 0 \leq i \leq I-1. \end{aligned}$$

The third term of the right hand side of the above expression gives the effect of the penalty cost, and its value can be comparatively large. But if we further assume that the repair time distribution is independent of the type of repair work, then the penalty term vanishes, and the property that $A(i) - (\sum_{k=1}^{K} p_k^{(i)}(C(i,k) + B(k)/q))$ is non-decreasing in i $(0 \leq i \leq I)$ can be shown to be sufficient. Other conditions are essentially the same as those in Theorem 2.5.

2.2. Extension

In this section we generalize the basic multi-repair-type maintenance model in the following manner: The operating cost of a machine depends on the type of repair work assuming it is repaired, as well as its operating condition. Also, the type of repair work on a machine to be repaired depends on the type of repair work which has been required on the machine if the repair decision had been chosen one period earlier. Define

$A(i,k)$: operating cost for a machine in the i-th operating condition which requires type k repair work if it is repaired.

$p_{k'}^{(i',k)}$: probability that a machine in the i'-th operating condition requires type k' repair if the repair decision is chosen given that the type k of repair has been required one period earlier.

This generalization seems to be reasonable when the previous example in Section 2.1 is considered. There, the age of a machine and its degree of deterioration correspond to the operating condition of a machine, and the type of repair work on it if repaired, respectively. It is reasonable to view the operating cost as depending on the degree of deterioration, and the degree of deterioration of a machine as depending on the degree of deterioration one period earlier.

Under this generalization, the following theorem holds:

Theorem 2.8. Assume the following conditions hold:

1. $B(k)$ is nonnegative and nondecreasing in k for $1 \le k \le K$.
2. $C(i,k)$ is nondecreasing in k $(1 \le k \le K)$ for each fixed i $(0 \le i \le I)$.

3. $C(i,k)$ is nondecreasing in i $(0 \leq i \leq I)$ for each fixed k $(1 \leq k \leq K)$.

4. $A(i,k)$ is nondecreasing in k $(1 \leq k \leq K)$ for each fixed i $(0 \leq i \leq I)$.

5. $A(i,k) - C(i,k)$ is nondecreasing in i $(0 \leq i \leq I)$ for each fixed k $(1 \leq k \leq K)$.

6. $P \geq C(0,1)$.

7. $q_k \geq q_{k+1} > 0$ for $1 \leq k \leq K-1$.

8. $P_i(\cdot) \subset P_{i+1}(\cdot)$ for $0 \leq i \leq I-1$.

9. $P^{(i',k)}(\cdot) \subset P^{(i',k+1)}(\cdot)$ for $1 \leq k \leq K-1$, $0 \leq i' \leq I$, where

$$P^{(i',k)}(k') = \sum_{j \leq k'} p_j^{(i',k)} .$$

10. $P^{(i',k)}(\cdot) \subset P^{(i'+1,k)}(\cdot)$ for $0 \leq i' \leq I-1$, $1 \leq k \leq K$.

Then there exists a stationary i control limit policy which minimizes the total expected α-discounted cost of this generalized multi-repair-type maintenance model. For the long-run expected average cost criterion, the eventual failure property on any machine is also needed.

Proof. The dynamic programming formulation of this generalized model is exactly the same as (2.1) if $A(i,k)$ and $p_{k'}^{(i',k)}$ replace $A(i)$ and $p_{k'}^{(i')}$ respectively. Lemma 2.1 still holds after these replacements, and Lemma 2.2 also holds if conditions 4 and 9 are added since they assure that $\langle V_\alpha(i,k,s_1,\ldots,s_K;n)\rangle_{1\text{-st}}$ is nondecreasing in k in the proof. It is clear that Lemma 2.3 holds if condition 6 of Lemma 2.3 is replaced

by condition 10 of Theorem 2.8. As in the proof of Lemma 2.4, we can show that $V_\alpha(i,k,s_1,\ldots,s_K;n)$ is nondecreasing in i $(0 \le i \le I)$ for each fixed $(k,s_1,\ldots,s_K) \in \mathcal{K} \times \mathcal{S}^S$, and for $n \ge 1$, if conditions 1 through 7 and 9 and 10 hold, and if $A(i,k)$ is nondecreasing in i $(0 \le i \le I)$ for each k $(1 \le k \le K)$. Finally it can be shown that $f_n(i,k,s_1,\ldots,s_K)$ is nondecreasing in i for $(k,s_1,\ldots,s_K) \in \mathcal{K} \times \mathcal{S}^S$, and for $n \ge 1$ if all the conditions hold, which yields the desired results. □

Condition 9 says that the heavier the current repair work required on a machine is, the heavier that its repair work will be in the future. Condition 10 means that the worse condition a machine is in, the more repair work required on it if it is repaired. None of the conditions in Theorem 2.8 are very restrictive conditions.

2.3. One Repairman for Each Repair Shop Case

We return to the basic multi-repair-type maintenance model. In this model, the repair work begins immediately on any machine sent to any type of repair shop. This is equivalent to saying that each repair shop has more than enough number of repairmen. In this section, we consider the other extreme case, where there is only one repairman in each type of repair facility. In this case, if a machine is to be repaired at some type of repair facility, it must wait until all the machines which have arrived earlier at the same type of repair facility are completely repaired. Machines waiting for repair form a queue in front of each type of repair facility. The system is schematically shown in Figure 2.3.

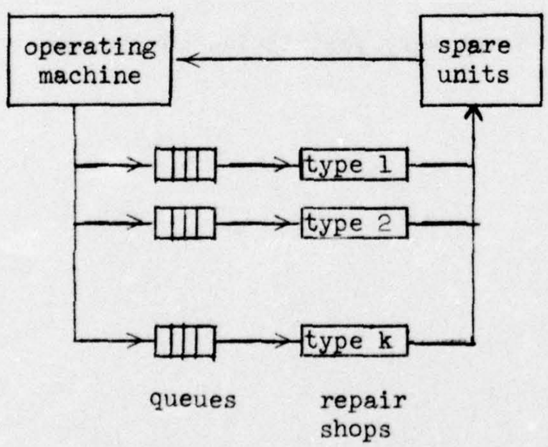

Figure 2.3. A Multi-Repair-Type System with One Repairman for Each Repair Facility.

Let q_k $(1 \leq k \leq K)$ be the probability that a machine in type k repair shop at the beginning of a period is completely repaired at the end of the period. Let $q_{ss'}^{(j)}$ be the probability that s' machines are still in the type j repair system at the end of the period given that s machines are in the type j repair system at the beginning of a period. Here the type k repair system includes the type k repair shop and the queue formed in front of the type k repair shop. Then

$$q^{(j)}_{s_j s_j} = \begin{cases} 1-q & \text{if } s'_j = s_j \text{ and } 1 \leq s_j \leq S+1 \\ q & \text{if } s'_j = s_j-1 \text{ and } 1 \leq s_j \leq S+1 \\ 0 & \text{if } s_j \neq s'_j \neq s_j-1 \text{ and } 1 \leq s_j \leq S+1 \text{ or } 0 = s_j \neq s'_j \\ 1 & \text{if } s_j = s'_j = 0. \end{cases}$$

The dynamic programming formulation of this model is the same as (2.1) where $q^{(j)}_{s_j s'_j}$'s are not binomial distributions as before but are given as above. As we can see from the proofs of Lemma 2.1 through Theorem 2.5, $q^{(j)}_{s_j s'_j}$'s explicitly appear only in the proof of Lemma 2.1. Hence, if Lemma 2.1 can be proved under this modified model, we can obtain the sufficient conditions for the optimality of a stationary i control limit policy by using the result of Theorem 2.5. Now in the proof of Lemma 2.1, it is sufficient to show that for $(i,k',s_1,\ldots,s_K) \in \mathcal{I} \times \mathcal{K} \times \mathcal{S}^{S-1} \cup \mathcal{S}^{S}_{0}$ and $n \geq 1$,

$$V_\alpha(i,k',s_1,\ldots,s_{k+1}+1,\ldots,s_K;n) \geq V_\alpha(i,k',s_1,\ldots,s_k+1,\ldots,s_K;n)$$

implies

$$R_\alpha(i,k',s_1,\ldots,s_{k+1}+1,\ldots,s_K;n) \geq R_\alpha(i,k',s_1,\ldots,s_k+1,\ldots,s_K;n),$$

and

$$V_\alpha(i,k',s_1,\ldots,s_k+1,\ldots,s_K;n) \geq V_\alpha(i,k',s_1,\ldots,s_k,\ldots,s_K;n)$$

implies

$$R_\alpha(i,k',s_1,\ldots,s_k+1,\ldots,s_K;n) \geq R_\alpha(i,k',s_1,\ldots,s_k,\ldots,s_K;n),$$

since the other arguments do not contain $q^{(j)}_{s_j s'_j}$'s explicitly. The

definition of R_α and the simple substitution of $q^{(j)}_{s_j s'_j}$ defined above give the desired result. As a conclusion of this section, we have Theorem 2.9.

Theorem 2.9. If all the conditions in Theorem 2.5 hold, then there exists a stationary i control limit policy which minimizes the total expected α-discounted cost of the multi-repair-type maintenance model with one repairman for each repair shop. For the long-run expected average cost criterion, the eventual failure property on any machine is also needed.

CHAPTER 3

MAINTENANCE MODELS WITH CONTROL OF QUEUE

In this chapter, maintenance models are treated in the context of control of queues. In these models, the decision maker has the option of opening or closing the repair shop when there are machines waiting for repair service, as well as the option of repairing or leaving an operating machine alone. A two-dimensional control limit policy is defined, and sufficient conditions for the optimality of a two-dimensional control limit policy are obtained for each model. Lastly, some computational remarks and conclusions are stated.

3.1. The Model and Some Examples

We consider the following discrete time Markov maintenance model, whose mechanism is illustrated in Fig. 3.1. There is an operating machine and S $(S \geq 1)$ identical spare machines in the system. At the beginning of each period, an operating machine is classified as being in one of $I+1$ $(I \geq 1)$ states as was done in the previous models. There is

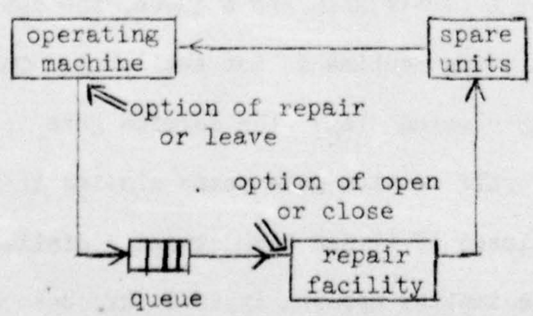

Figure 3.1. A Machine Maintenance System with Control of Queue.

a repair shop in the system, and an operating machine can be sent to the repair shop for the repair work at any period. We assume that there is only one repairman in the repair shop, or only one machine can be repaired in the repair shop during any time period. Therefore a machine sent to the repair shop must wait until all the machines which have already arrived at the repair shop are completely repaired. Waiting machines form a queue. No waiting is necessary if there has been no machine in the repair shop.

At the beginning of each period, just after the observation of the state of the system, the decision maker has the option of opening or closing the repair shop, as well as the option of repairing or leaving an operating machine alone. Therefore at each period, four actions are available. They are denoted as a_{LC}, a_{LO}, a_{RC}, and a_{RO} respectively, where L, R, C and O mean to leave an operating machine in operation, to repair an operating machine, to close the repair service gate, and to open the repair service gate respectively. For example, a_{LC} is the action: leave an operating machine in operation and close the service gate. If there is no operating machine, or if all the machines are in the repair system consisting of a repair shop and a queue, the option of repairing or leaving an operating machine is not available. Only the option of opening (a_O) or closing (a_C) the service gate is then available. Closing the repair service gate means closing it if it has been open and keeping it closed if it has been closed. Similarly, opening the repair service gate implies opening it if it has been closed and keeping it open if it has been open. The repair work can be performed

only when the service gate is open. If a repair job on a machine is interrupted by the decision to close the gate, the rest of the repair work is postponed until the gate reopens. Repaired machines are available as spare units.

If the decision to leave an operating machine in operation is chosen, it keeps operating and its state evolves from i to j in one period according to the transition probability $p_{ij} \geq 0$. If the decision to repair an operating machine is selected, it is immediately sent to the repair system, and is instantly replaced by a spare unit, if any are available. The new operating machine begins to operate just after replacement in its best condition.

The costs associated with the system are:

$A(i)$: operating cost for a machine in the i-th operating condition $(0 \leq i \leq I)$ per period.

$C(i)$: material cost for repairing a machine in the i-th operating condition $(0 \leq i \leq I)$.

$K(s,k)$: holding cost per period of s $(0 \leq s \leq S+1)$ machines in the repair system when the gate is closed $(k = 0)$ or open $(k = 1)$ at the beginning of the period.

E : set up cost of opening a closed repair shop.

F : shut down cost of closing an open repair shop.

G : service cost of operating an open repair shop per period.

P : penalty cost assessed per period while no operating machine is available.

The objective function is the total expected α-discounted cost, and the structure of an optimal policy minimizing such a criterion is studied.

Before proceeding further, we give some examples to clarify the applicability of this model in the practical world.

The first example is an airplane repair problem for a privately owned flying shcool. Suppose the instructor in the flying school owns two airplanes and a repair shop for repairing them. He teaches flying using one airplane at a time. He inspects the condition of the airplane in service periodically, and he classifies it as being in one of a finite number of states. He continues using the same airplane, until he judges that it should be repaired. Then the other airplane, if available, replaces it and begins to operate in its best condition. The previously used airplane is sent to the repair shop, which is either open or closed. Repair work can be performed only when the repair shop is open, and a repaired airplane will be ready for future use in its best condition. Keeping the repair shop always open may not be economical since he must pay salary to a repairman, and other costs to keep it open, even when no repair work is needed. Keeping the repair shop closed too long is also undesirable since it may prevent airplanes from being available, thereby resulting in a loss of revenue. The problem of finding the best schedule for hiring a repairman and for replacing an operating airplane can be formulated as a model to be studied in this chapter.

Another example pertains to cleaning suits. Suppose a person has several suits. He wears one suit continuously until he finds that it requires cleaning, and then he changes the suits using one remaining in his wardrobe. Assume the degree of cleanliness of a suit is

observable. The cleaner the suit he wears the better the impression he gives, and hence, the more comfortable he himself becomes. In this sense, the cost of wearing a suit is associated with its cleanliness. When he decides to change the suit to the clean one, it does not necessarily follow that he immediately sends the used suit for dry-cleaning since such an action may be laborious and time consuming. However, too infrequent visits to the cleaners may lead to the situation where he has no clean suits available. Assuming that a suit is available in its cleanest condition after dry-cleaning, the problem of determining the best schedule for suit changes and laundry visits can be formulated as a model to be studied.

3.2. Control Limit Policy with Respect to Operating Machine

In the last section, we did not specify the repair time distribution. It is clear, though, if the repair service gate is closed, no repair work can be performed. When the repair service gate is open, we assume the following in this section: The number of machines in the repair system solely determines the distribution of the number of machines in the repair system one period later. Let $q_{ss'}$ be the probability that s' machines are still in the repair system at the end of the period, given that s machines are in the repair system at the beginning of a period.

In order to describe the model, we need to specify the number of machines in the repair system and the state of the repair service

gate, as well as the condition of an operating machine. Let $V_\alpha^k(i,s;n)$ be the minimum expected n period α-discounted cost given that the operating machine is in the i-th operating condition, the number of machines in the repair system is s, and the state of the repair service gate is k ($k = 0$ means the gate is closed, and $k = 1$ means it is open) at the beginning. Then by letting $V_\alpha^k(i,s;n) = 0$ for any feasible i, s, and k, $V_\alpha^k(i,s;n)$ $(n \geq 1)$ satisfies a set of recursive equations:

For $0 \leq i \leq I, 0 \leq s \leq S$,

$$V_\alpha^0(i,s;n) = \min \begin{pmatrix} A(i) + K(s,0) + \alpha \sum_{j=0}^{I} p_{ij} V_\alpha^0(j,s;n-1), \\ A(i) + K(s,0) + E + G + \alpha \sum_{j=0}^{I} p_{ij} \sum_{s'=0}^{s} q_{ss'} V_\alpha^1(j,s';n-1), \\ C(i) + K(s,0) + \alpha \sum_{j=0}^{I} p_{0j} V_\alpha^0(j,s+1;n-1), \\ C(i) + K(s,0) + E + G + \alpha \sum_{j=0}^{I} p_{0j} \sum_{s'=0}^{s+1} q_{s+1,s'} V_\alpha^1(j,s';n-1) \end{pmatrix},$$

$$V_\alpha^1(i,s;n) = \min \begin{pmatrix} A(i) + K(s,1) + F + \alpha \sum_{j=0}^{I} p_{ij} V_\alpha^0(j,s;n-1), \\ A(i) + K(s,1) + G + \alpha \sum_{j=0}^{I} p_{ij} \sum_{s'=0}^{s} q_{ss'} V_\alpha^1(j,s';n-1), \\ C(i) + K(s,1) + F + \alpha \sum_{j=0}^{I} p_{0j} V_\alpha^0(j,s+1;n-1), \\ C(i) + K(s,1) + G + \alpha \sum_{j=0}^{I} p_{0j} \sum_{s'=0}^{s+1} q_{s+1,s'} V_\alpha^1(j,s';n-1) \end{pmatrix},$$

$$V_\alpha^0(i,S+1;n)$$
$$= \min \begin{pmatrix} P + K(S+1,0) + \alpha \sum_{j=0}^{I} p_{0j} V_\alpha^0(j,S+1;n-1), \\ P + K(S+1,0) + E + G + \alpha \sum_{j=0}^{I} p_{0j} \sum_{s'=0}^{S+1} q_{S+1,s'} V_\alpha^1(j,s';n-1) \end{pmatrix},$$

$$V_\alpha^1(i,S+1;n)$$
$$= \min \begin{pmatrix} P + K(S+1,1) + F + \alpha \sum_{j=0}^{I} p_{0j} V_\alpha^0(j,S+1;n-1), \\ P + K(S+1,1) + G + \alpha \sum_{j=0}^{I} p_{0j} \sum_{s'=0}^{S+1} q_{S+1,s'} V_\alpha^1(j,s';n-1) \end{pmatrix}. \tag{3.1}$$

Note that $\langle V_\alpha^k(i,s;n)\rangle_{1\text{-st}}$, $\langle V_\alpha^k(i,s;n)\rangle_{2\text{-nd}}$, $\langle V_\alpha^k(i,s;n)\rangle_{3\text{-rd}}$, and $\langle V_\alpha^k(i,s;n)\rangle_{4\text{-th}}$ are the n period costs of taking a_{LC}, a_{LO}, a_{RC}, and a_{RO} respectively at the beginning followed by the best policy. When $s = S+1$, a_C and a_O are the only available actions. Also note that the i in $V_\alpha^k(i,S+1;n)$ is a dummy, and is used just for notational simplicity. When all the machines are in the repair system, no operating machine is available, and i in the above expression has no meaning.

As the system is a Markov decision process with discount factor $0 \leq \alpha < 1$, the existence of a stationary policy minimizing the total α-discounted cost is guaranteed. For further discussion on the structure of an optimal policy, the following simplification is useful. Let

$$R_\alpha(i,s;n) = \alpha \sum_{j=0}^{I} p_{ij}\, V_\alpha^0(j,s;n)$$

$$Q_\alpha(i,s;n) = \alpha \sum_{j=0}^{I} p_{ij} \sum_{s'=0}^{s} q_{ss'}\, V_\alpha^1(j,s';n) \; . \tag{3.2}$$

Then for $n \geq 1$, $0 \leq i \leq I$, $0 \leq s \leq S$, (3.1) can be simplified to

$$\begin{aligned} &V_\alpha^0(i,s;n) \\ &\quad = \min\{A(i) + K(s,0) + R_\alpha(i,s;n-1),\ A(i) + K(s,0) + E + G \\ &\qquad + Q_\alpha(i,s;n-1),\ C(i) + K(s,0) + R_\alpha(0,s+1;n-1),\ C(i) \\ &\qquad + K(s,0) + E + G + Q_\alpha(0,s+1;n-1)\} \; , \end{aligned}$$

$$\begin{aligned} &V_\alpha^1(i,s;n) \\ &\quad = \min\{A(i) + K(s,1) + F + R_\alpha(i,s;n-1),\ A(i) + K(s,1) + G \\ &\qquad + Q_\alpha(i,s;n-1),\ C(i) + K(s,1) + F + R_\alpha(0,s+1;n-1), \\ &\qquad C(i) + K(s,1) + G + Q_\alpha(0,s+1;n-1)\} \; , \end{aligned}$$

$$\begin{aligned} &V_\alpha^0(i,S+1;n) \\ &\quad = \min\{P + K(S+1,0) + R_\alpha(0,S+1;n-1),\ P + K(S+1,0) + E + G + Q_\alpha(0,S+1;n-1)\} , \end{aligned}$$

$$\begin{aligned} &V_\alpha^1(i,S+1;n) \\ &\quad = \min\{P + K(S+1,1) + F + R_\alpha(0,S+1;n-1), P + K(S+1,1) + G + Q_\alpha(0,S+1;n-1)\} \; . \end{aligned} \tag{3.3}$$

The problem is now to find the structure of an optimal policy. It is conceivable that an optimal policy has the form that the repair

decision is taken if and only if the condition of an operating machine becomes worse than some critical value, and that the decision to open the repair service gate is taken if and only if the number of machines in the repair system exceeds some critical value. The following types of control limit policies will then be suitable candidates as the class of policies satisfied by an optimal policy.

Definition. A control limit policy with respect to an operating machine is a nonrandomized policy where, as the option of repairing or leaving an operating machine is concerned, there is an i for each k $(k = 0,1)$, s $(0 \leq s \leq S)$, and n $(n \geq 1)$, say $i_{k,s,n}$, called the control limit, such that for all (i,k,s) with $i < i_{k,s,n}$, the decision at period n is to leave it in operation, and for all (i,k,s) with $i \geq i_{k,s,n}$, the decision at period n is to repair it. A control limit policy with respect to a repair shop is a nonrandomized policy where, as the option of opening or closing the repair service gate is concerned, there is an s for each k $(k = 0, 1)$, i $(0 \leq i \leq I)$, and n $(n \geq 1)$, say $s_{k,i,n}$, called the control limit, such that for all (i,k,s) with $s < s_{k,i,n}$, the decision at period n is to close it, and for all (i,k,s) with $s \geq s_{k,i,n}$, the decision at period n is to open it. A two-dimensional control limit policy is a control limit policy with respect to both an operating machine and a repair shop.

The rest of this section concentrates on finding sufficient conditions for the optimality of a control limit policy with respect to an operating machine. The following lemma holds:

Lemma 3.1. Assume the following conditions hold:

1. $A(i)$ is nondecreasing in i for $0 \leq i \leq I$.

2. $C(i)$ is nondecreasing in i for $0 \leq i \leq I$.

3. $P_i(\cdot) \subset P_{i+1}(\cdot)$ for $0 \leq i \leq I-1$.

Then for each fixed s $(0 \leq s \leq S)$, k $(k = 0,1)$, and $n \geq 1$, $V_\alpha^k(i,s,n)$ is nondecreasing in i $(0 \leq i \leq I)$.

Proof. Mathematical induction is applied. For $n = 1$, for each s $(0 \leq s \leq S)$ and k $(k = 0,1)$, $V_\alpha^k(i,s;1)$ is clearly nondecreasing in i $(0 \leq i \leq I)$, by 1 and 2. Suppose the assertion holds for $n = m-1 \geq 1$. Then by the definitions of R_α and Q_α, both $R_\alpha(i,s;m-1)$ and $Q_\alpha(i,s;m-1)$ become nondecreasing in i $(0 \leq i \leq I)$ because of 3. Hence, all the terms on the right hand sides of both $V_\alpha^0(i,s;m)$ and $V_\alpha^1(i,s;m)$ are nondecreasing in i $(0 \leq i \leq I)$ if 1 and 2 are satisfied. which completes the mathematical induction and the proof. □

Next theorem assures the optimality of a stationary control limit policy with respect to an operating machine.

Theorem 3.2. If conditions 2 and 3 of Lemma 3.1 hold, and if

4. $A(i) - C(i)$ is nondecreasing in i for $0 \leq i \leq I$,

then there exists a stationary control limit policy with respect to an operating machine which minimizes the total expected α-discounted cost of the maintenance model with control of queue.

Proof. We first consider the n-stage problem. For $n \geq 1$, $0 \leq i \leq I$, $0 \leq s \leq S$, and $k = 0,1$, let

$$f^k_{L,n}(i,s) = \min\{\langle V^k_\alpha(i,s;n)\rangle_{1\text{-st}},\ \langle V^k_\alpha(i,s;n)\rangle_{2\text{-nd}}\}$$

$$f^k_{R,n}(i,s) = \min\{\langle V^k_\alpha(i,s;n)\rangle_{3\text{-rd}},\ \langle V^k_\alpha(i,s;n)\rangle_{4\text{-th}}\} .$$

Then $f^k_{L,n}(i,s)$ can be interpreted as the minimum n-stage α-discounted cost given that a machine is in the i-th operating condition, the number of machines in the repair system is s, and the state of the repair service gate is k at the beginning, and only the decision to keep a machine in operation is allowed at the beginning. If only the decision to repair an operating machine is allowed at the beginning, we have $f^k_{R,n}(i,s)$.

Now, for $0 \leq i \leq I$, $0 \leq s \leq S$, and $n \geq 0$,

$$\begin{aligned} & f^0_{L,n+1}(i,s) - f^0_{R,n+1}(i,s) \\ & \quad = A(i) + K(s,0) + \min\{R_\alpha(i,s;n),\ E + G + Q_\alpha(i,s;n)\} \\ & \qquad - C(i) - K(s,0) - \min\{R_\alpha(0,s+1;n),\ E + G + Q_\alpha(0,s+1;n)\} . \end{aligned}$$

As conditions 2 and 4 of Theorem 3.2 imply condition 1 of Lemma 3.1, Lemma 3.1 holds, which shows that both $R_\alpha(i,s;n)$ and $Q_\alpha(i,s;n)$ are nondecreasing in i $(0 \leq i \leq I)$. With this, and by 3, we have that $f^0_{L,n}(i,s) - f^0_{R,n}(i,s)$ is nondecreasing in i $(0 \leq i \leq I)$ for each s $(0 \leq s \leq S)$ and $n \geq 1$. Similarly, it can be easily shown that $f^1_{L,n}(i,s) - f^1_{R,n}(i,s)$ is nondecreasing in i $(0 \leq i \leq I)$ for each s

$(0 \leq s \leq S)$ and $n \geq 1$. Hence, there exists a set of critical numbers $i^n_{k,s}$ $(k = 0,1,\ 0 \leq s \leq S)$ for each $n \geq 1$ such that

$$f^k_{L,n}(i,s) < f^k_{R,n}(i,s) \quad \text{for any} \quad i < i^n_{k,s}$$

and

$$f^k_{L,n}(i,s) \geq f^k_{R,n}(i,s) \quad \text{for any} \quad i \geq i^n_{k,s} \ .$$

Therefore, as far as the option of repairing or leaving an operating machine is concerned, at the beginning of each n-stage problem, if the state of the system is (i,s,k), to leave a machine in operation is optimal if its operating condition i is less than $i^n_{k,s}$, and to repair it is optimal if i is greater than or equal to $i^n_{k,s}$, which is a control limit policy with respect to an operating machine. For the infinite horizon problem, the technique of Theorem 2.2 of [5] can be applied here to show the optimality of a stationary control limit policy with respect to an operating machine. □

At the end of this section, we make a few remarks on the relations between optimal decisions when the repair gate is closed and the corresponding optimal decisions when the gate is open. For this discussion, we assume that E, F are nonnegative.

<u>Lemma 3.3.</u> When the repair service gate is closed at the beginning of a period, if a_{LO} (a_{RO}) is optimal for some (i,s), then a_{LO} $(a_{RO}$, respectively) is also optimal for the same (i,s) when the gate is open.

Proof. We prove the statement for a_{LO}. In a similar fashion, the statement for a_{RO} can be proved.

Let

$$Q_\alpha(i,s) = \lim_{n\to\infty} Q_\alpha(i,s;n)$$

$$R_\alpha(i,s) = \lim_{n\to\infty} R_\alpha(i,s;n) \ .$$

For a fixed (i,s), and $k = 0$, a_{LO} being better than a_{LC} implies

$$A(i) + K(s,0) + E + G + Q_\alpha(i,s) \leq A(i) + K(s,0) + R_\alpha(i,s) \ .$$

Hence,

$$\begin{aligned} A(i) + K(s,1) + G + Q_\alpha(i,s) &\leq A(i) + K(s,1) + E + G + Q_\alpha(i,s) \\ &\leq A(i) + K(s,1) + R_\alpha(i,s) \\ &\leq A(i) + K(s,1) + F + R_\alpha(i,s). \end{aligned}$$

Thus, for (i,s) and $k = 1$, a_{LO} is better than a_{LC}. Similarly, the statement that a_{LO} is better than a_{RC} for $k = 0$ implies that a_{LO} is better than a_{RC} for $k = 1$, and the statement that a_{LO} is better than a_{RO} for $k = 0$ implies that a_{LO} is better than a_{RO} for $k = 1$, yielding that for (i,s), a_{LO} is optimal when the gate is open. □

Lemma 3.4. When the repair service gate is open at the beginning of a period, if a_{RC} (a_{LC}) is optimal for some (i,s), then a_{RC} (a_{LC}, respectively) is also optimal for the same (i,s) when the gate is closed.

Proof. Similar to Lemma 3.3, and hence can be omitted. □

The same argument still holds for the case where there is no operating machine. In that case, if opening the gate is optimal when the gate is closed, then keeping it open is optimal when the gate is open. Conversely, if closing the gate is optimal when the gate is open, then keeping it closed is optimal when the gate is closed.

3.3. Case Where Repair Time is Negligible

In this section sufficient conditions to ensure the existence of a control limit policy with respect to a repair shop minimizing the total α-discounted cost are of interest. The following assumption is made on the repair time of a machine throughout this section. For $0 \leq s \leq S+1$,

$$q_{ss'} = \begin{cases} 1 & \text{if } s' = 0 \\ 0 & \text{if } s' \neq 0 \end{cases}$$

The above assumption implies that the repair time of each machine is negligible compared with the length of a period, and hence, all the machines brought into the repair system can be repaired completely in one period when the service gate is open. This assumption will be reasonable if purchasing or ordering machines takes place instead of repairing them when the "gate is open."

We first show the following lemma:

Lemma 3.5. Assume the following conditions hold:

1. $C(i)$ is nondecreasing in i for $0 \leq i \leq I$.
2. $K(s,k)$ is nondecreasing in s $(0 \leq s \leq S+1)$ for $k = 0,1$.

3. $P \geq C(I)$.

Then $V_\alpha^k(i,s;n)$ is nondecreasing in s $(0 \leq s \leq S+1)$ for each $0 \leq i \leq I$, $k = 0,1$, and $n \geq 1$.

Proof. Proof is by mathematical induction. For $n = 1$, $0 \leq i \leq I$, and $0 \leq s \leq S$,

$$V_\alpha^0(i,s;1) = \min\{A(i) + K(s,0),\ C(i) + K(s,0)\},$$

which is nondecreasing in s $(0 \leq s \leq S)$ by 2. Also

$$\begin{aligned} V_\alpha^0(i,S+1;1) &- V_\alpha^0(i,S;1) \\ &= P + K(S+1,0) - (\min\{A(i),\ C(i)\} + K(S,0)) \\ &\geq P - \min\{A(i),\ C(i)\}, \quad \text{by 2} \\ &\geq P - C(i) \geq 0, \quad \text{by 1 and 3.} \end{aligned}$$

Hence, $V_\alpha^0(i,s;1)$ is nondecreasing in s for $0 \leq s \leq S+1$. In a similar manner, it is shown that $V_\alpha^1(i,s;1)$ is nondecreasing in s for $0 \leq s \leq S+1$. Suppose the hypothesis holds for $n = m-1 \geq 1$. Then for $0 \leq s \leq S$,

$$\begin{aligned} V_\alpha^0&(i,s;m) \\ &= K(s,0) + \min\{A(i) + R_\alpha(i,s;m-1),\ A(i) + E + G + Q_\alpha(i,s;m-1), \\ &\qquad C(i) + R_\alpha(0,s+1;m-1),\ C(i) + E + G + Q_\alpha(0,s+1,m-1)\}. \end{aligned}$$

Now both $R_\alpha(i,s;m-1)$ and $R_\alpha(0,s+1;m-1)$ are nondecreasing in s $(0 \leq s \leq S)$ by the induction hypothesis and by the definition of R_α, and $Q_\alpha(i,s;m-1)$ is constant in s since

$$Q_\alpha(i,s,m-1) = \alpha \sum_{j=0}^{I} p_{ij} V_\alpha^1(j,0;m-1) .$$

Also $K(s,0)$ is nondecreasing in s $(0 \le s \le S)$, yielding that $V_\alpha^0(i,s;m)$ is nondecreasing in s $(0 \le s \le S)$. Also,

$$\begin{aligned}
&V_\alpha^0(i,S+1;m) - V_\alpha^0(i,S;m) \\
&\quad \ge P + K(S+1,0) + \min\{R_\alpha(0,S+1;m-1),\ E+G+Q_\alpha(0,S+1;m-1)\} \\
&\qquad - (C(i) + K(S,0) + \min\{R_\alpha(0,S+1;m-1),\ E+G+Q_\alpha(0,S+1;m-1)\}) \\
&\quad = P - C(i) + K(S+1,0) - K(S,0) \ge 0, \text{ by 1, 2 and 3.}
\end{aligned}$$

Thus, $V_\alpha^0(i,s;m)$ is nondecreasing in s $(0 \le s \le S+1)$ for each fixed $0 \le i \le I$. Similarly, we can show that $V_\alpha^1(i,s;m)$ is nondecreasing in s $(0 \le s \le S+1)$, completing the mathematical induction and the proof. □

Using the above lemma and the simplfying assumption on $q_{ss'}$, we can prove the following theorem, which gives sufficient conditions for the optimality of a control limit policy with respect to a repair shop.

Theorem 3.6. If all the conditions in Lemma 3.5 hold, then there exists a stationary control limit policy with respect to a repair shop which minimizes the total expected α-discounted cost of the simplified maintenance model with control of queue.

<u>Proof</u>. For $n \geq 1$, $0 \leq i \leq I$, $0 \leq s \leq S$, and $k = 0,1$, let

$$f^k_{C,n}(i,s) = \min\{\langle V^k_\alpha(i,s;n)\rangle_{1\text{-st}},\ \langle V^k_\alpha(i,s;n)\rangle_{3\text{-rd}}\}$$

$$f^k_{O,n}(i,s) = \min\{\langle V^k_\alpha(i,s;n)\rangle_{2\text{-nd}},\ \langle V^k_\alpha(i,s;n)\rangle_{4\text{-th}}\} .$$

$f^k_{C,n}(i,s)$ can be interpreted as the minimum n-stage α-discounted cost given that the state of the system is (i,s,k) at the beginning, and only the decision to close the repair shop is allowed at the beginning. If only the decision to open the repair shop is allowed at the beginning, we have $f^k_{O,n}(i,s)$.

The proof of this theorem is similar to that of Theorem 3.2, and it is sufficient to verify that $f^k_{C,n}(i,s) - f^k_{O,n}(i,s)$ is nondecreasing in s $(0 \leq s \leq S)$ for each fixed i $(0 \leq i \leq I)$, k $(k = 0,1)$, and $n \geq 1$. But, for $n \geq 0$, $0 \leq i \leq I$, and $k = 0,1$,

$$\begin{aligned} & f^k_{C,n+1}(i,s) - f^k_{O,n+1}(i,s) \\ &\quad = \min\{A(i) + R_\alpha(i,s;n),\ C(i) + R_\alpha(0,s+1;n)\} \\ &\qquad - \min\{A(i) + E + G + Q_\alpha(i,s;n),\ C(i) + E + G + Q_\alpha(0,s+1;n)\} , \end{aligned}$$

and both $R_\alpha(i,s;n)$ and $R_\alpha(0,s+1;n)$ are nondecreasing in s $(0 \leq s \leq S)$ by Lemma 3.5, and both $Q_\alpha(i,s;n)$ and $Q_\alpha(0,s+1;n)$ are constant in s by the simplifying assumption on $q_{ss'}$. Hence, $f^0_{C,n}(i,s) - f^0_{O,n}(i,s)$ is nondecreasing in s $(0 \leq s \leq S)$ for $n \geq 1$ and $0 \leq i \leq I$. In a similar manner, $f^1_{C,n}(i,s) - f^1_{O,n}(i,s)$ is shown to be nondecreasing in s $(0 \leq s \leq S)$, which is what we need. $\square$

Combining Theorems 3.2 and 3.6 gives sufficient conditions under which a two-dimensional control limit policy is optimal.

Theorem 3.7. Assume the following conditions hold:

1. $C(i)$ is nondecreasing in i for $0 \leq i \leq I$.
2. $A(i) - C(i)$ is nondecreasing in i for $0 \leq i \leq I$.
3. $K(s,k)$ is nondecreasing in s $(0 \leq s \leq S+1)$ for $k = 0,1$.
4. $P \geq C(I)$.
5. $P_i(\cdot) \subset P_{i+1}(\cdot)$ for $0 \leq i \leq I-1$.

Then there exists a stationary two-dimensional control limit policy minimizing the total expected α-discounted cost of the simplified maintenance model with control of queue.

One realization of an optimal stationary two-dimensional control limit policy is illustrated in Fig. 3.2. As previously pointed out in Lemmas 3.3 and 3.4, the region where a_{LO} (a_{RO}) is optimal, called the optimal region of a_{LO} (a_{RO}, respectively), when the gate is open covers the optimal region of a_{LO} (a_{RO}, respectively) when the gate is closed.

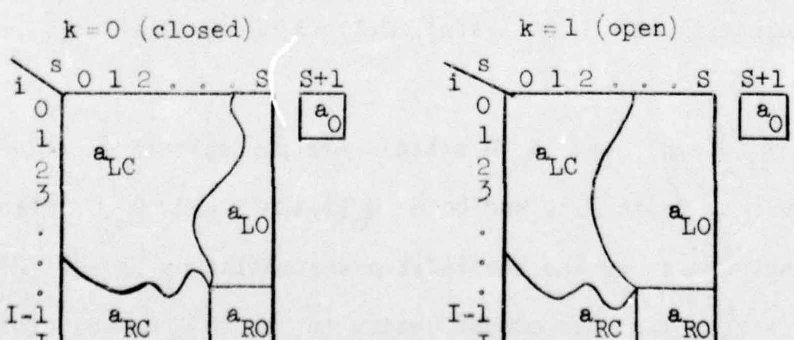

Figure 3.2. A Typical Optimal Two-Dimensional Control Limit Policy

Further, the optimal region of a_{LC} (a_{RC}) when the gate is closed covers that of a_{LC} (a_{RC}, respectively) when the gate is open. That also implies that the boundary of two optimal regions of a_{LO} and a_{RO} for $k = 0$ is also a boundary of the corresponding optimal regions for $k = 1$, and that the boundary of two optimal regions of a_{LC} and a_{RC} for $k = 1$ is also a boundary of the corresponding optimal regions for $k = 0$.

Notice that for both $k = 0$ and $k = 1$ cases, the boundary of optimal regions of a_{RO} and a_{RC} is vertical, and that of a_{RO} and a_{LO} is horizontal. This can be seen by comparing the appropriate terms in (3.3). For example, to verify that the boundary of optimal regions of a_{RO} and a_{RC} is vertical for $k = 0$, it is enough to see that the direction of the inequality

$$C(i) + K(s,0) + R_{\alpha}(0,s+1;n) \lesseqgtr C(i) + K(s,0) + E + G + Q_{\alpha}(0,s+1;n)$$

does not depend on i for each fixed s $(0 \leq s \leq S)$.

In the theory of control of the service process by turning on and off servers, there is a class of policies called hysteresis loop policies which are optimal under a wide class of cost functions. The form of the policy is: provide no service if the system size is m or less, and when the system size increases to M $(M > m)$, turn the server on and continue serving until the system size again drops to m.

In this model we notice that the control limit on the action of closing (a_{LC} or a_{RC}) or opening (a_{LO} or a_{RO}) the repair service gate for $k = 0$ is no less than the corresponding control limit for $k = 1$, keeping the condition of an operating machine fixed.

Thus, if we keep the condition i of an operating machine fixed, an optimal policy has the following form: keep the gate closed (a_{LC} or a_{RC} is taken) if the number of machines waiting for repair service is m_i or less, and when the number of machines waiting for repair service increases to M_i ($M_i > m_i$), open the gate (a_{LO} or a_{RO} is taken), and keep it open until the number of machines to be repaired again drops to m_i. This is a hysteresis loop policy. However, the actual sample paths or trajectories are unlikely to be hysteretic since the operating condition of an operating machine may change as time passes.

3.4. Case Where Repair Time Distribution is Geometric

In order to assure the optimality of a two-dimensional control limit policy, we assumed in the last section that the repair time of a machine was negligibly small as compared with the length of a period. This assumption is relaxed in this section at the cost of optimality of a two-dimensional control limit policy. Here we assume that the repair time of a machine when the repair service gate is open has a geometric distribution with parameter q. Then,

$$q_{ss'} = \begin{cases} q & \text{if } s' = s-1,\ 1 \leq s \leq S+1 \\ 1-q & \text{if } s' = s,\ 1 \leq s \leq S+1 \\ 0 & \text{if } s \neq s' \neq s-1,\ 1 \leq s \leq S+1, \text{ or if } 0 = s \neq s' \\ 1 & \text{if } s = s' = 0, \end{cases}$$

since at most one machine can be repaired in the repair shop at each period no matter how many machines are in the repair system.

Consider a stationary control limit policy with respect to an operating machine. The existence of such a policy minimizing the total α-discounted cost is guaranteed if the conditions in Theorem 3.2 are all satisfied. In the case of a stationary control limit policy with respect to a repair shop, the analysis becomes much more complicated. In some papers, a convex analysis or a piecewise convex analysis has been suggested for this type of model, but it is unrealistic to expect our cost criterion to be convex, and a piecewise convex analysis does not seem to work for discrete time models. Therefore, the analysis must be performed without assuming a nice structure on the cost criterion. A bounding technique used in [5] then seems to be the only candidate for the analysis of this type of model.

Lemma 3.8. Assume the following conditions hold:

1. $A(i)$ is nondecreasing in i for $0 \leq i \leq I$.
2. $C(i)$ is nondecreasing in i for $0 \leq i \leq I$.
3. $K(s,k)$ is nondecreasing in s $(0 \leq s \leq S+1)$ for $k = 0,1$.
4. $P \geq \min\{A(0), C(0)\}$.
5. $P_i(\cdot) \subset P_{i+1}(\cdot)$ for $0 \leq i \leq I-1$.

Then, for $1 \leq s \leq S+1$, $0 \leq i \leq I$, $k = 0,1$, and $n \geq 1$,

$$V_\alpha^k(i,s;n) - V_\alpha^k(i,s-1;n) \leq \bar{M}_n ,$$

where

$$\bar{M}_n = \frac{1-\alpha^n}{1-\alpha} (P - \min\{A(0), C(0)\} + \bar{K}) ,$$

and

$$\bar{K} = \max_{s,k'} \{K(s,k') - K(s-1,k')\} .$$

Proof. Mathematical induction is applied. For $n = 1$, it is easy to see that for $0 \le i \le I$, $1 \le s \le S+1$, $k = 0,1$,

$$V_\alpha^k(i,s;1) - V_\alpha^k(i,s-1;1) \le P - \min\{A(0), C(0)\} + \bar{K} = \bar{M}_1 .$$

Suppose the argument holds for $n = m-1 \ge 1$, and consider the case for $n = m$. For $k = 0$ and $1 \le s \le S$, we compare the corresponding terms of the right-hand side of (3.3).

$$\begin{aligned}
&\langle V_\alpha^0(i,s;m)\rangle_{1\text{-st}} - \langle V_\alpha^0(i,s-1;m)\rangle_{1\text{-st}} \\
&= A(i) + K(s,0) + R_\alpha(i,s;m-1) - (A(i) + K(s-1,0) + R_\alpha(i,s-1;m-1)) \\
&= K(s,0) - K(s-1,0) + \alpha \sum_{j=0}^{I} p_{ij}(V_\alpha^0(j,s;m-1) - V_\alpha^0(j,s-1;m-1) \\
&\le K(s,0) - K(s-1,0) + \alpha \sum_{j=0}^{I} p_{ij}\, \bar{M}_{m-1} \le \bar{K} + \alpha\bar{M}_{m-1} .
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\langle V_\alpha^0(i,s;m)\rangle_{2\text{-nd}} - \langle V_\alpha^0(i,s-1;m)\rangle_{2\text{-nd}} \\
&= A(i) + K(s,0) + E + G + Q_\alpha(i,s;m-1) - (A(i) + K(s-1,0) + E + G + Q_\alpha(i,s-1;m-1)) \\
&= \begin{cases}
K(s,0) - K(s-1,0) + \alpha \displaystyle\sum_{j=0}^{I} p_{ij}[(1-q)(V_\alpha^1(j,s;m-1) - V_\alpha^1(j,s-1;m-1)) \\
\qquad\qquad + q(V_\alpha^1(j,s-1;m-1) - V_\alpha^1(j,s-2;m-2))] \\
\qquad\qquad \text{for } 2 \le s \le S \\
K(s,0) - K(s-1,0) + \alpha \displaystyle\sum_{j=0}^{I} p_{ij}(1-q)(V_\alpha^1(j,1;m-1) - V_\alpha^1(j,0;m-1)) \\
\qquad\qquad \text{for } s = 1
\end{cases} \\
&\le \bar{K} + \alpha \sum_{j=0}^{I} p_{ij}\, \bar{M}_{m-1} = \bar{K} + \alpha\bar{M}_{m-1} .
\end{aligned}$$

In a similar manner, the comparison of the corresponding third terms and that of the fourth terms yield the same upper bound $\bar{K} + \alpha\bar{M}_{m-1}$. Hence, for $1 \leq s \leq S$, $0 \leq i \leq I$,

$$V_\alpha^0(i,s;m) - V_\alpha^0(i,s-1;m) \leq \bar{K} + \alpha\bar{M}_{m-1} .$$

For $s = S+1$, and $0 \leq i \leq I$,

$$\begin{aligned}
&\langle V_\alpha^0(i,S+1;m)\rangle_{1\text{-st}} - \langle V_\alpha^0(i,S;m)\rangle_{1\text{-st}} \\
&= P + K(S+1,0) + R_\alpha(0,S+1;m-1) - (A(i) + K(S,0) + R_\alpha(i,S;m-1)) \\
&= P - A(i) + K(S+1,0) - K(S,0) \\
&\qquad + \alpha\, [\sum_{j=0}^{I} p_{0j}\, V_\alpha^0(j,S+1;m-1) - \sum_{j=0}^{I} p_{ij}\, V_\alpha^0(j,S;m-1)] \\
&\leq P - A(i) + K(S+1,0) - K(S,0) \\
&\qquad + \alpha \sum_{j=0}^{I} p_{0j}(V_\alpha^0(j,S+1;m-1) - V_\alpha^0(j,S;m-1)) \qquad \text{by 5 and Lemma 3.1} \\
&\leq P - \min\{A(0),\ C(0)\} + \bar{K} + \alpha\bar{M}_{m-1} .
\end{aligned}$$

Similarly, using $P_0(\cdot) \subset P_i(\cdot)$ and that $V_\alpha^1(j,s;m-1)$ is nondecreasing in j $(0 \leq j \leq I)$, $\langle V_\alpha^0(i,S+1;m)\rangle_{2\text{-nd}} - \langle V_\alpha^0(i,S;m)\rangle_{2\text{-nd}}$ can be shown to have the same upper bound. Also,

$$\begin{aligned}
&\langle V_\alpha^0(i,S+1,m)\rangle_{1\text{-st}} - \langle V_\alpha^0(i,S;m)\rangle_{3\text{-rd}} \\
&= P + K(S+1,0) + R_\alpha(0,S+1;m) - (C(i) + K(S,0) + R_\alpha(0,S+1;m-1)) \\
&\leq P - \min\{A(0),\ C(0)\} + \bar{K} .
\end{aligned}$$

We can show the same upper bound also on $\langle V_\alpha^0(i,S+1;m)\rangle_{2\text{-nd}} - \langle V_\alpha^0(i,S;m)\rangle_{4\text{-th}}$, yielding that for $0 \leq i \leq I$,

$$V_\alpha^0(i,S+1;m) - V_\alpha^0(i,S;m) \leq P - \min\{A(0),\ C(0)\} + \bar{K} + \alpha\bar{M}_{m-1} .$$

As $P \geq \min\{A(0),\ C(0)\}$ from condition 4, we have for $0 \leq i \leq I$, and $1 \leq s \leq S+1$,

$$V_\alpha^0(i,s;m) - V_\alpha^0(i,s-1;m) \leq P - \min\{A(0),\ C(0)\} + \bar{K} + \alpha\bar{M}_{m-1} = \bar{M}_m .$$

A similar argument indicates that for $0 \leq i \leq I$, and $1 \leq s \leq S+1$,

$$V_\alpha^1(i,s;m) - V_\alpha^1(i,s-1;m) \leq \bar{M}_m ,$$

completing the mathematical induction, and hence the proof. □

For the future use, let

$$\bar{M}_\infty = \lim_{n\to\infty} \bar{M}_n = \frac{1}{1-\alpha}\,(P - \min\{A(0),\ C(0)\} + \bar{K}).$$

Before deriving a lower bound on the same expression, we make the following remark: If $\{x_n\}$ is a sequence with the following recursive relation starting from $x_0 = 0$,

$$x_n = \min\{a + \alpha x_{n-1},\ b\} \quad \text{for} \quad n \geq 1,$$

then we have for $0 \leq \alpha < 1$,

$$x_n = \min\{\frac{1-\alpha^n}{1-\alpha}\,a,\ b\} \qquad \text{for} \quad n \geq 1 .$$

Lemma 3.9. If $K(s,k)$ is nondecreasing in s $(0 \leq s \leq S+1)$ for $k = 0,1$, then for $1 \leq s \leq S+1$, $0 \leq i \leq I$, $k = 0,1$, and $n \geq 1$,

$$V_\alpha^k(i,s;n) - V_\alpha^k(i,s-1;n) \geq \underline{M}_n ,$$

where

$$\underline{M}_n = \min\{\underline{K} \frac{1 - (\alpha(1-q))^n}{1 - \alpha(1-q)} , P - C(I) + \min_{k'} \{K(S+1,k') - K(S,k')\}\}$$

and

$$\underline{K} = \min_{s,k'} \{K(s,k') - K(s-1,k')\} .$$

Proof. Proof is again by mathematical induction. For $n = 1$, $0 \leq i \leq I$, and $1 \leq s \leq S$,

$$V_\alpha^0(i,s;1) - V_\alpha^0(i,s-1;1) = K(s,0) - K(s-1,0) \geq \underline{K} .$$

And,

$$\begin{aligned} & V_\alpha^0(i,S+1;1) - V_\alpha^0(i,S;1) \\ & \quad = P - \min\{A(i), C(i)\} + K(S+1,0) - K(S,0) \\ & \quad \geq P - C(I) + \min_{k'}\{K(S+1,k') - K(S,k')\} . \end{aligned}$$

Hence, for $k = 0$, the hypothesis is true for $n = 1$. By a similar argument, the hypothesis is shown to hold for $k = 1$ and $n = 1$. Suppose the hypothesis is true for $n = m-1 \geq 1$, and consider the case for $n = m$. We compare the corresponding terms as in the previous lemma. Fix i $(0 \leq i \leq I)$. For $1 \leq s \leq S$,

$$\langle V_\alpha^0(i,s;m)\rangle_{1\text{-st}} - \langle V_\alpha^0(i,s-1;m)\rangle_{1\text{-st}}$$

$$= K(s,0) - K(s-1,0) + \alpha \sum_{j=0}^{I} p_{ij}[V_\alpha^0(j,s;m-1) - V_\alpha^0(j,s-1;m-1)]$$

$$\geq K(s,0) - K(s-1,0) + \alpha \sum_{j=0}^{I} p_{ij}\, \underline{M}_{m-1} \geq \underline{K} + \alpha\underline{M}_{m-1} .$$

For $2 \leq s \leq S$,

$$\langle V_\alpha^0(i,s;m)\rangle_{2\text{-nd}} - \langle V_\alpha^0(i,s-1;m)\rangle_{2\text{-nd}}$$

$$= K(s,0) - K(s-1,0) + \alpha \sum_{j=0}^{I} p_{ij}[(1-q)(V_\alpha^1(j,s;m-1) - V_\alpha^1(j,s-1;m-1)) + q(V_\alpha^1(j,s-1;m-1) - V_\alpha^1(j,s-2;m-1))]$$

$$\geq \underline{K} + \alpha \sum_{j=0}^{I} p_{ij}((1-q)\underline{M}_{m-1} + q\underline{M}_{m-1}) = \underline{K} + \alpha\underline{M}_{m-1} .$$

For $s = 1$,

$$\langle V_\alpha^0(i,1;m)\rangle_{2\text{-nd}} - \langle V_\alpha^0(i,0;m)\rangle_{2\text{-nd}}$$

$$= K(1,0) - K(0,0) + \alpha \sum_{j=0}^{I} p_{ij}[(1-q)V_\alpha^1(j,1;m-1) + qV_\alpha^1(j,0;m-1) - V_\alpha^1(j,0;m-1)]$$

$$\geq \underline{K} + \alpha(1-q)\underline{M}_{m-1} .$$

Comparison of the corresponding third terms and that of the fourth terms can be performed in a similar manner, yielding that for $1 \leq s \leq S$,

$$V_\alpha^0(i,s;m) - V_\alpha^0(i,s-1;m) \geq \underline{K} + \alpha(1-q)\underline{M}_{m-1} .$$

For $s = S+1$,

$$V_\alpha^0(i,S+1;m) - V_\alpha^0(i,S;m)$$

$$\geq P + K(S+1,0) + \min\{\alpha \sum_{j=0}^{I} p_{0j} V_\alpha^0(j,S+1;m-1),\ E + G$$

$$+ \alpha \sum_{j=0}^{I} p_{0j} \sum_{s'=0}^{S+1} q_{S+1,s'} V_\alpha^1(j,s';m-1)\}$$

$$- [C(i) + K(S,0) + \min\{\alpha \sum_{j=0}^{I} p_{0j} V_\alpha^0(j,S+1;m-1),\ E + G$$

$$+ \alpha \sum_{j=0}^{I} p_{0j} \sum_{s'=0}^{S+1} q_{S+1,s'} V_\alpha^1(j,s';m-1)\}]$$

$$= P - C(i) + K(S+1,0) - K(S,0)$$

$$\geq P - C(I) + K(S+1,0) - K(S,0)\ .$$

Therefore, for $1 \leq s \leq S+1$, and $0 \leq i \leq I$,

$$V_\alpha^0(i,s;m) - V_\alpha^0(i,s-1;m) \geq \min\{\underline{K} + \alpha(1-q)\underline{M}_{m-1},\ P - C(I) + K(S+1,0) - K(S,0)\}.$$

A similar argument yields for $1 \leq s \leq S+1$ and $0 \leq i \leq I$,

$$V_\alpha^1(i,s;m) - V_\alpha^1(i,s-1;m) \geq \min\{\underline{K} + \alpha(1-q)\underline{M}_{m-1},\ P - C(I) + K(S+1,1) - K(S,1)\}\ .$$

Therefore, for $1 \leq s \leq S+1$, $0 \leq i \leq I$, and $k = 0,1$,

$$V_\alpha^k(i,s;m) - V_\alpha^k(i,s-1;m)$$

$$\geq \min\{\underline{K} + \alpha(1-q)\underline{M}_{m-1},\ P - C(I) + \min_{k'}\{K(S+1,k') - K(S,k')\}\}$$

$$= \min\{\underline{K}\,\frac{1-(\alpha(1-q))^n}{1-\alpha(1-q)},\ P - C(I) + \min_{k'}\{K(S+1,k') - K(S,k')\}\}$$

$$= \underline{M}_m\ ,$$

which completes the mathematical induction. □

Analogous to $\bar{M}_\infty$, we define

$$\underline{M}_\infty = \lim_{n \to \infty} \underline{M}_n = \min\{\frac{\underline{K}}{1 - \alpha(1-q)} , P - C(I) + \min_{k'}\{K(S+1,k') - K(S,k')\}\} .$$

It is easy to see that

$$\bar{M}_n - \underline{M}_n \leq \bar{M}_{n+1} - \underline{M}_{n+1} \leq \bar{M}_\infty - \underline{M}_\infty .$$

<u>Lemma 3.10</u>. For $0 \leq i \leq I$, $0 \leq s \leq S+1$, and $n \geq 1$,

$$V^0_\alpha(i,s;n) - V^1_\alpha(i,s;n) = K(s,0) - K(s,1) + \delta_n(i,s)$$

where $-F \leq \delta_n(i,s) \leq E$.

<u>Proof</u>. For $0 \leq s \leq S+1$, $0 \leq i \leq I$, and $n \geq 1$,

$$\langle V^0_\alpha(i,s;n)\rangle_{j\text{-th}} - \langle V^1_\alpha(i,s;n)\rangle_{j\text{-th}} = K(s,0) - K(s,1) - F ,$$

for $j = 1$ and 3 ($j = 1$ only if $s = S+1$) ,

$$\langle V^0_\alpha(i,s;n)\rangle_{j\text{-th}} - \langle V^1_\alpha(i,s;n)\rangle_{j\text{-th}} = K(s,0) - K(s,1) + E ,$$

for $j = 2$ and 4 ($j = 2$ only if $s = S+1$).

Therefore,

$$K(s,0)-K(s,1)-F \leq V^0_\alpha(i,s;n) - V^1_\alpha(i,s;n) \leq K(s,0) - K(s,1) + E ,$$

which implies

$$-F \leq \delta_n(i,s) \equiv V^0_\alpha(i,s;n) - V^1_\alpha(i,s;n) - (K(s,0) - K(s,1)) \leq E . \quad \square$$

In this section we have failed to derive sufficient conditions for the optimality of a control limit policy with respect to a repair shop. However, it is possible to obtain sufficient conditions under which a control limit type of property holds between two actions a_{LC} and a_{LO}, and between a_{RC} and a_{RO}. The next lemma is useful for this purpose.

Lemma 3.11. Assume all the conditions of Lemma 3.8 hold, and furthermore, assume the following condition holds:

6. $K(s+1,0) - K(s,0) \geq K(s+1,1) - K(s,1) + E + F + q(\bar{M}_\infty - \underline{M}_\infty)$, $0 \leq s \leq S$.

Then, for $0 \leq s \leq S+1$, $0 \leq i \leq I$, and $n \geq 1$, $R_\alpha(i,s;n) - Q_\alpha(i,s;n)$ is nondecreasing in s.

Proof. For $1 \leq s \leq S+1$, $0 \leq i \leq I$, and $n \geq 1$,

$$R_\alpha(i,s;n) - Q_\alpha(i,s;n)$$

$$= \alpha \sum_{j=0}^{I} p_{ij}[V_\alpha^0(j,s;n) - \sum_{s'=0}^{s} q_{ss'} V_\alpha^1(j,s';n)]$$

$$= \alpha \sum_{j=0}^{I} p_{ij}[K(s,0) - K(s,1) + \delta_n(j,s) + V_\alpha^1(j,s;n)$$

$$- (1-q)V_\alpha^1(j,s;n) - qV_\alpha^1(j,s-1;n)], \text{ by Lemma 3.10}$$

$$= \alpha \sum_{j=0}^{I} p_{ij}[K(s,0) - K(s,1) + \delta_n(j,s) + q[V_\alpha^1(j,s;n) - V_\alpha^1(j,s-1;n)]].$$

Hence, for $1 \leq s \leq S$, $0 \leq i \leq I$, and $n \geq 1$,

$$(R_\alpha(i,s+1;n) - Q_\alpha(i,s+1;n)) - (R_\alpha(i,s;n) - Q_\alpha(i,s;n))$$

$$= \alpha \sum_{j=0}^{I} p_{ij}[K(s+1,0) - K(s+1,1) + \delta_n(j,s+1) + q(V_\alpha^1(j,s+1;n) - V_\alpha^1(j,s;n))$$

$$- [K(s,0) - K(s,1) + \delta_n(j,s) + q(V_\alpha^1(j,s;n) - V_\alpha^1(j,s-1;n))]]$$

$$\geq \alpha \sum_{j=0}^{I} p_{ij}[K(s+1,0) - K(s+1,1) - (K(s,0) - K(s,1)) - E - F + q(\underline{M}_n - \bar{M}_n)] ,$$

by Lemmas 3.8, 3.9, and 3.10

≥ 0, by 6 and since $\underline{M}_n - \bar{M}_n \geq \underline{M}_\infty - \bar{M}_\infty$.

For $s = 0$, $0 \leq i \leq I$, and $n \geq 1$,

$$R_\alpha(i,0;n) - Q_\alpha(i,0;n) = \alpha \sum_{j=0}^{I} p_{ij}[V_\alpha^0(j,0;n) - V_\alpha^1(j,0;n)] .$$

Hence,

$$(R_\alpha(i,1;n) - Q_\alpha(i,1;n)) - (R_\alpha(i,0;n) - Q_\alpha(i,0;n))$$

$$= \alpha \sum_{j=0}^{I} p_{ij}[K(1,0) - K(1,1) + \delta_n(j,1) + q(V_\alpha^1(j,1;n) - V_\alpha^1(j,0;n))$$

$$- V_\alpha^0(j,0;n) + V_\alpha^1(j,0;n)]$$

$$= \alpha \sum_{j=0}^{I} p_{ij}[K(1,0) - K(1,1) - (K(0,0) - K(0,1)) + \delta_n(j,1)$$

$$- \delta_n(j,0) + q(V_\alpha^1(j,1;n) - V_\alpha^1(j,0;n))]$$

$$\geq \alpha \sum_{j=0}^{I} p_{ij}[K(1,0) - K(1,1) - (K(0,0) - K(0,1)) - F - E + q\underline{M}_n] ,$$

by Lemmas 3.9 and 3.10

≥ 0, by condition 6 and since $\underline{M}_n \geq 0$,

which completes the proof. □

Notice that if the conditions of Lemma 3.11 are all satisfied, then for $0 \le i \le I$, $R_\alpha(i,s) - Q_\alpha(i,s)$ is also nondecreasing in s for $0 \le s \le S+1$.

Suppose that all the conditions in both Lemmas 3.2 and 3.11 are satisfied. By Lemma 3.2, there is a stationary control limit policy with respect to an operating machine which minimizes the total expected α-discounted cost. Consider the structure of the optimal policy further, and we find that its i-s diagram, as is seen in Fig. 3.3 is divided into upper and lower divisions for each k. The action of leaving an operating machine in operation is taken in each state in the upper region, and the action of repairing an operating machine is taken in each state in the lower region. The former has two alternatives a_{LO} and a_{LC}, while the latter has two alternatives a_{RO} and a_{RC}.

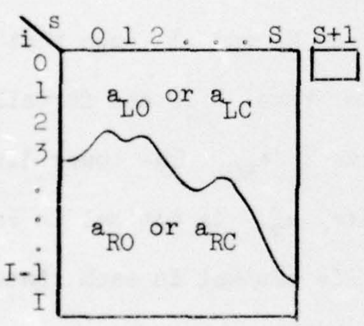

Figure 3.3. A Typical Optimal Control Limit Policy with Respect to an Operating Machine.

We now focus on the possibility of subdividing each region having two alternatives. When the state of the system at the beginning is (i,s,k) with $k = 0$, $0 \le i \le I$, and $0 \le s \le S$, the difference between the total cost of choosing a_{LC} at the beginning followed by the best policy and that of choosing a_{LO} at the beginning followed by the best policy is

$$A(i) + K(s,0) + R_\alpha(i,s) - (A(i) + K(s,0) + E + G + Q_\alpha(i,s))$$
$$= R_\alpha(i,s) - Q_\alpha(i,s) - (E + G) ,$$

which is nondecreasing in s $(0 \leq s \leq S)$ by Lemma 3.11. For $k = 1$, the corresponding difference becomes $R_\alpha(i,s) - Q_\alpha(i,s) + (F - G)$, which is also nondecreasing in s $(0 \leq s \leq S)$. Therefore, there exist critical numbers $s^L_{i,k}$ for each fixed k $(k = 0,1)$ and i $(0 \leq i \leq I)$ such that for all (i,s,k) with $s < s^L_{i,k}$, a_{LC} is better than a_{LO}, and for all (i,s,k) with $s \geq s^L_{i,k}$, a_{LO} is no worse than a_{LC}. This implies that the upper division where a_{LO} or a_{LC} is optimal can be divided into left and right subdivisions. a_{LC} is optimal in each state in the left subdivision, while a_{LO} is optimal in each state in the right subdivision. In a similar manner, we can show that there exist critical numbers $s^R_{i,k}$ for each fixed k and i such that for all (i,s,k) with $s < s^R_{i,k}$, a_{RC} is better than a_{RO}, and for all (i,s,k) with $s \geq s^R_{i,k}$, a_{RO} is no worse than a_{RC}. The lower division can be divided into two subdivisions. There, a_{RC} is optimal in each state in the left subdivision, while a_{RO} is optimal in each state in the right subdivision. We call this type of policy a stationary <u>two-dimensional weak control limit policy</u>. One realization of a two-dimensional weak control limit policy, optimizing our problem, is shown in Fig. 3.4.

The control limits found in this kind of policy are those on the action of repairing or leaving an operating machine, those on the action of a_{LC} or a_{LO}, and those on the action of a_{RC} or a_{RO}. Control limits on the action of opening or closing the repair shop might not exist. In this sense, this type of policy is weaker than a two-dimensional control limit policy.

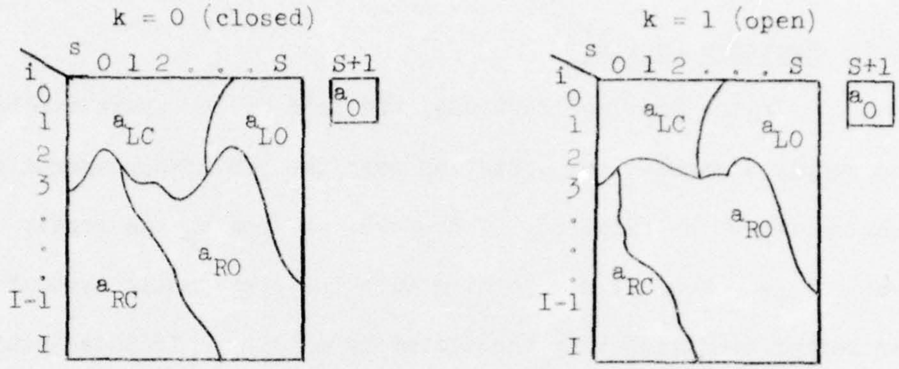

Figure 3.4. A Typical Optimal Two-Dimensional Weak Control Limit Policy.

As a conclusion of this section, we restate the above discussion as a theorem.

Theorem 3.12. Assume the following conditions hold:

1. $C(i)$ is nondecreasing in i for $0 \le i \le I$.
2. $A(i) - C(i)$ is nondecreasing in i for $0 \le i \le I$.
3. $K(s,k)$ is nondecreasing in s $(0 \le s \le S+1)$ for each $k = 0,1$.
4. $P \ge \min\{A(0), C(0)\}$.
5. $P_i(\cdot) \subset P_{i+1}(\cdot)$ for $0 \le i \le I-1$.
6. $K(s+1,0) - K(s,0) \ge K(s+1,1) - K(s,1) + E + F + q(\bar{M}_\infty - \underline{M}_\infty)$, where

$$\bar{M}_\infty = \frac{1}{1-\alpha}\left[P - \min\{A(0), C(0)\} + \bar{K}\right]$$

$$\underline{M}_\infty = \min\left\{\frac{\underline{K}}{1-\alpha(1-q)}\, ,\; P - C(I) + \min_{k'}\{K(S+1,k') - K(S,k')\}\right\}$$

$$\bar{K} = \max_{s,k'}\{K(s,k') - K(s-1,k')\}$$

$$\underline{K} = \min_{s,k'}\{K(s,k') - K(s-1,k')\}\,.$$

Then there exists a stationary two-dimensional weak control limit policy which minimizes the total expected α-discounted cost of the model.

3.5. Exchange Options

In the previous sections, the role of the spare machines was to supply a machine for operation when the previously operating machine was chosen to be repaired. Therefore, as long as the repair decision was not selected, we did nothing with the spare units even if they were in better condition than the operating machine. In this section, the action space is expanded to allow for the case of exchanging an operating machine with one of the available spare units at any time period so that the operating machine can always be in better condition than any spare unit. The option available at any period is to exchange an operating machine with one of the available spare units, or to send an operating machine to the repair system and introduce one of the available spare units as a new operating machine. A decision to keep an operating machine in operation is considered as a decision to exchange an operating machine with itself. Machines sent to the repair system are repaired in one period when the repair service gate is open, and repaired machines are available as spare units in their best conditions. The cost structures are the same as before. An exchange action may take place at no extra cost. The structure of an optimal policy minimizing the total expected α-discounted cost is of interest.

In order to describe the model, we must specify the number of spare machines in the i-th condition for each $0 \leq i \leq I$, since spare units are no longer guaranteed to be in the best condition. Let $w = (i,s,s_0,\ldots,s_I,k)$ denote the state of the system where there is an operating machine in the i-th condition, the number of machines in the

repair system is s, the number of spare units in the j-th condition is s_j $(0 \le j \le I)$, and the state of the repair service gate is k. Necessarily,

$$0 \le i \le I,\ s \ge 0,\ s_j \ge 0\ (0 \le j \le I),\ s + \sum_{j=0}^{I} s_j = S.$$

We denote $(-,S+1,0,0,\ldots,0,0)$ to represent the state where no operating machine is available.

For $w = (i,s,s_0,\ldots,s_I,k)$, let

$$U(w) = \{j \mid s_j \ge 1,\ 0 \le j \le I\} .$$

There is at least one spare unit in the j-th condition if $j \in U(w)$.

Define the following actions:

$a_{E,j}$: exchange an operating machine for a spare unit in the j-th condition.

$a_{R,j}$: send an operating machine to the repair system and introduce a spare unit in the j-th condition as a new operating machine.

$a_{R,-}$: send an operating machine to the repair system.

a_C (a_O) : close (open, respectively) the repair service gate.

The action space $A(w)$ of this system depends on the state of the system $w = (i,s,s_0,\ldots,s_I,k)$, and can be represented as

$$A(w) = A^1(w) \times A^2(w) ,$$

where for $U(w) \ne \emptyset$, $A^1(w) = \{a_{E,j}, a_{R,j'} \mid j \in U(w) \cup \{i\},\ j' \in U(w)\}$, and for $U(w) = \emptyset$ and $s = S$, $A^1(w) = \{a_{E,i}, a_{R,-}\}$, and $A'(w) = \emptyset$ if $s = S+1$, and $A^2(w) = \{a_C, a_O\}$.

Let $V_\alpha^k(i,s,s_0,\ldots,s_I;n)$ be the minimum n period expected α-discounted cost starting from state $(i,s,s_0,\ldots,s_I,k)$. Then we have the following recursive equation:

For $U(w) \neq \emptyset$, $0 \leq i \leq I$, $0 \leq s \leq S$, and $n \geq 0$,

$$V_\alpha^0(i,s,s_0,\ldots,s_j,\ldots,s_I;n+1)$$

$$= \min\{ \min_{j\in U(w)\cup\{i\}} \{A(i) + K(s,0) + \alpha \sum_{j'=0}^{I} p_{jj'} \cdot V_\alpha^0(j',s,s_0,\ldots,s_j-1,\ldots,s_i+1,\ldots,s_I;n)\},$$

$$\min_{j\in U(w)} \{C(i) + K(s,0) + \alpha \sum_{j'=0}^{I} p_{jj'} \cdot V_\alpha^0(j',s+1,s_0,\ldots,s_j-1,\ldots,s_I;n)\},$$

$$\min_{j\in U(w)\cup\{i\}} \{A(i) + K(s,0) + E + G + \alpha \sum_{j'=0}^{I} p_{jj'} \cdot V_\alpha^1(j',0,s_0+s,s_1,\ldots,s_j-1,\ldots,s_i+1,\ldots,s_I;n)\},$$

$$\min_{j\in U(w)} \{C(i) + K(s,0) + E + G + \alpha \sum_{j'=0}^{I} p_{jj'} \cdot V_\alpha^1(j',0,s_0+s+1,s_1,\ldots,s_j-1,\ldots,s_i,\ldots,s_I,n)\}\}.$$

$$V_\alpha^1(i,s,s_0,\ldots,s_j,\ldots,s_I;n+1)$$

$$= \min\{ \min_{j\in U(w)\cup\{i\}} \{A(i) + K(s,1) + F + \alpha \sum_{j'=0}^{I} p_{jj'} \cdot V_\alpha^0(j',s,s_0,\ldots,s_j-1,\ldots,s_i+1,\ldots,s_I;n)\},$$

$$\min_{j\in U(w)} \{C(i) + K(s,1) + F + \alpha \sum_{j'=0}^{I} p_{jj'}$$

$$\cdot V_\alpha^0(j',s+1,s_0,\ldots,s_j-1,\ldots,s_I;n)\} ,$$

$$\min_{j\in U(w)\cup\{i\}} \{A(i) + K(s,1) + G + \alpha \sum_{j'=0}^{I} p_{jj'}$$

$$\cdot V_\alpha^1(j',0,s_0+s,s_1,\ldots,s_j-1,\ldots,s_i+1,\ldots,s_I;n)\} ,$$

$$\min_{j\in U(w)} \{C(i) + K(s,1) + G + \alpha \sum_{j'=0}^{I} p_{jj'}$$

$$\cdot V_\alpha^1(j',0,s_0+s+1,s_1,\ldots,s_{j-1},\ldots,s_i,\ldots,s_I;n)\}\} .$$

For $U(w) = \emptyset$, $s = S$, $0 \le i \le I$, and $n \ge 0$,

$$V_\alpha^0(i,S,0,\ldots,0;n+1)$$

$$= \min\{A(i) + K(S,0) + \alpha \sum_{j'=0}^{I} p_{ij'} V_\alpha^0(j',S,0,\ldots,0;n) ,$$

$$C(i) + K(S,0) + \alpha V_\alpha^0(-,S+1,0,\ldots,0;n),$$

$$A(i) + K(S,0) + E + G + \alpha \sum_{j'=0}^{I} p_{ij'} V_\alpha^1(j',0,S,0,\ldots,0;n),$$

$$C(i) + K(S,0) + E + G + \alpha V_\alpha^1(0,0,S,0,\ldots,0;n)\} .$$

$$V_\alpha^1(i,S,0,\ldots,0;n+1)$$

$$= \min\{A(i) + K(S,1) + F + \alpha \sum_{j'=0}^{I} p_{ij'} V_\alpha^0(j',S,0,\ldots,0;n),$$

$$C(i) + K(S,1) + F + \alpha V_\alpha^0(-,S+1,0,\ldots,0;n) ,$$

$$A(i) + K(S,1) + G + \alpha \sum_{j'=0}^{I} p_{ij'} V_\alpha^1(j',0,S,0,\ldots,0;n) ,$$

$$C(i) + K(S,1) + G + \alpha V_\alpha^1(0,0,S,0,\ldots,0;n)\} .$$

For $U(w) = \emptyset$, $s = S+1$, and $n \geq 0$,

$$V_\alpha^0(-,S+1,0,\ldots,0;n+1) = \min\{P + K(S+1,0) + \alpha V_\alpha^0(-,S+1,0,\ldots,0;n),$$
$$P + K(S+1,0) + E + G + \alpha V_\alpha^1(0,0,S,0,\ldots,0;n)\} .$$

$$V_\alpha^1(-,S+1,0,\ldots,0;n+1) = \min\{ P + K(S+1,1) + F + \alpha V_\alpha^0(-,S+1,0,\ldots,0;n),$$
$$P + K(S+1,1) + G + \alpha V_\alpha^1(0,0,S,0,\ldots,0;n)\} . \quad (3.4)$$

This model is much more complicated than the previous models without the exchange options. Fortunately, the analysis on the structure of an optimal policy can be performed in a similar manner.

<u>Lemma 3.13</u>. Assume the following conditions hold:

1. $C(i)$ is nondecreasing in i for $0 \leq i \leq I$.
2. $A(i)$ is nondecreasing in i for $0 \leq i \leq I$.
3. $P_i(\cdot) \subset P_{i+1}(\cdot)$ for $0 \leq i \leq I-1$.

Then for $n \geq 1$, and $k = 0,1$,

(a) $V_\alpha^k(i,s,s_0,\ldots,s_j+1,\ldots,s_I;n)$ is nondecreasing in j $(0 \le j \le I)$ for $0 \le i \le I$, and for each fixed $(s,s_0,\ldots,s_I)$ with $s \ge 0$, $s_h \ge 0$ $(0 \le h \le I)$ and $s + \sum_{h=0}^{I} s_h = S-1$.

(b) $V_\alpha^k(i,s,s_0,\ldots,s_I;n)$ is nondecreasing in i $(0 \le i \le I)$ for each fixed $(s,s_0,\ldots,s_I)$ with $s \ge 0$, $s_h \ge 0$ $(0 \le h \le I)$ and $s + \sum_{h=0}^{I} s_h = S$.

Proof. Mathematical induction is applied simultaneously on both (a) and (b). Both (a) and (b) hold for $n = 1$. Suppose the claims (a) and (b) both hold for $n = m \ge 1$. Then from (3.4), for $w = (i,s,s_0,\ldots,s_j,\ldots,s_I,0)$,

$$\langle V_\alpha^0(i,s,s_0,\ldots,s_j+1,\ldots,s_I;m+1)\rangle_{1\text{-st}}$$

$$= A(i) + K(s,0) + \min\{\alpha \sum_{j'=0}^{I} p_{jj'}\, V_\alpha^0(j',s,s_0,\ldots,s_j,\ldots,s_i+1,\ldots,s_I;m),$$

$$\min_{h\in U(w)\cup\{i\}} \{\alpha \sum_{j'=0}^{I} p_{hj'}\, V_\alpha^0(j',s,s_0,\ldots,s_h-1,\ldots,s_j+1,$$

$$\ldots,s_i+1,\ldots,s_I;m)\}\} .$$

Now by the inductive hypothesis on (b) for $n = m$, and by condition 3, $\sum_{j'=0}^{I} p_{jj'}\, V_\alpha^0(j',s,s_0,\ldots,s_j,\ldots,s_i+1,\ldots,s_I;m)$ is nondecreasing in j $(0 \le j \le I)$. Also by the inductive hypothesis on (a) for $n = m$, $V_\alpha^0(j',s,s_0,\ldots,s_h-1,\ldots,s_j+1,\ldots,s_i+1,\ldots,s_I;m)$ is nondecreasing in j $(0 \le j \le I)$. Therefore, the first term of $V_\alpha^0(i,s,s_0,\ldots,s_j+1,\ldots,s_I;m+1)$ is nondecreasing in j $(0 \le j \le I)$. In a similar manner, other terms are shown to be nondecreasing in j $(0 \le j \le I)$, yielding

that $V_\alpha^0(i,s,s_0,\dots,s_j+1,\dots,s_I;m+1)$ is nondecreasing in j $(0 \le j \le I)$. The same argument follows for $V_\alpha^1(i,s,s_0,\dots,s_j+1,\dots,s_I;m+1)$, so that (a) holds for $n = m+1$.

For $U(w) \neq \emptyset$ with $w = (i,s,s_0,\dots,s_I,0)$,

$$\langle V_\alpha^0(i,s,s_0,\dots,s_I;m+1)\rangle_{1\text{-st}}$$

$$= A(i) + K(s,0) + \min\{ \min_{j\in U(w)} \alpha \sum_{j'=0}^{I} p_{jj'}\, V_\alpha^0(j',s,s_0,\dots,s_j-1,\dots,s_i+1, \dots,s_I,m),$$

$$\alpha \sum_{j'=0}^{I} p_{ij'}\, V_\alpha^0(j',s,s_0,\dots,s_i,\dots,s_I;m)\} .$$

By the inductive assumption on (a) for $n = m$, $V_\alpha^0(j,s,s_0,\dots,s_j-1, \dots,s_i+1,\dots,s_I,m)$ is nondecreasing in i, and by the inductive assumption on (b) for $n = m$ and by condition 3, $\sum_{j'=0}^{I} p_{ij'}\, V_\alpha^0(j',s, s_0,\dots,s_I;m)$ is nondecreasing in i $(0 \le i \le I)$. Hence, the first term of $V_\alpha^0(i,s,s_0,\dots,s_I;m+1)$ is nondecreasing in i $(0 \le i \le I)$, using condition 2. The same argument follows for the third term. For the second and the fourth terms, notice that they depend on i only through $C(i)$, and hence, they are nondecreasing in i $(0 \le i \le I)$ by condition 1. Therefore, $V_\alpha^0(i,s,s_0,\dots,s_I;m+1)$ is nondecreasing in i $(0 \le i \le I)$ for $U(w) \neq \emptyset$. In a similar manner, $V_\alpha^1(i,s,s_0,\dots,s_I;m+1)$ can be shown to be nondecreasing in i $(0 \le i \le I)$ for $U(w) \neq \emptyset$. For $U(w) = \emptyset$, the only possibility to be considered is the case $(s,s_0,\dots,s_I) = (S,0,\dots,0)$, and it is

clear that $V_\alpha^k(i,S,0,\ldots,0;m+1)$ is nondecreasing in i $(0 \le i \le I)$ for $k = 0,1$ from (3.4). Therefore, (b) also holds for $n = m+1$, completing the mathematical induction and the proof. □

Lemma 3.14. Assume the following conditions hold:

1. $K(s,k)$ is nondecreasing in s $(0 \le s \le S+1)$ for $k = 0,1$.
2. $P \ge C(I)$.

Then for $k = 0,1$, $0 \le i \le I$, $1 \le m \le S$, $0 \le s \le m-1$, $\sum_{j=1}^{I} s_j = S-m$, $s_j \ge 0$ $(0 \le j \le I)$, and $n \ge 1$,

$$V_\alpha^k(i,s,m-s,s_1,\ldots,s_I;n) \le V_\alpha^k(i,s+1,m-s-1,s_1,\ldots,s_I;n).$$

Proof. Mathematical induction is again applied. From (3.4), it is clear that the claim holds for $n = 1$. Suppose it holds for n (≥ 1), and consider the case for $n+1$. Comparisons of the corresponding terms of the right-hand side of (3.4) will give the desired result. For example, we compare the corresponding second terms. Let $w = (i,s,m-s,s_1,\ldots,s_I,0)$, and let $w' = (i,s+1,m-s-1,s_1,\ldots,s_I,0)$. Then for $1 \le m \le S$, $0 \le s \le m-2$, and for $1 \le m \le S-1$, $s = m-1$,

$$\langle V_\alpha^0(i,s,m-s,s_1,\ldots,s_I;n+1)\rangle_{2\text{-nd}}$$

$$= \min_{j\in U(w)} \{C(i) + K(s,0) + \alpha \sum_{j'=0}^{I} p_{jj'} V_\alpha^0(j',s+1,m-s,s_1,\ldots,s_j-1,\ldots,s_I;n)\}$$

$$\le \min_{j\in U(w')} \{C(i) + K(s+1,0) + \alpha \sum_{j'=0}^{I} p_{jj'} V_\alpha^0(j',s+2,m-s-1,s_1,\ldots,s_j-1,\ldots,s_I;n)\},$$

by condition 1, inductive hypothesis, and $U(w') \subset U(w)$

$$= \langle V_\alpha^0(i,s+1,m-s-1,s_1,\ldots,s_I;n+1)\rangle_{2\text{-nd}} .$$

Notice that if conditions 4 and 5 are satisfied,

$$V_\alpha^0(-,S+1,0,\ldots,0;n) - V_\alpha^0(i,S,0,\ldots,0;n) \geq P - C(i) + K(S+1,0) - K(S,0) \geq 0,$$

for any $0 \leq i \leq I$. Therefore, for $m = S$, $s = S-1$, and $0 \leq i \leq I$,

$$\begin{aligned}
&\langle V_\alpha^0(i,S-1,1,0,\ldots,0;n+1)\rangle_{2\text{-nd}} \\
&\quad = C(i) + K(S-1,0) + \alpha \sum_{j'=0}^{I} p_{0j'} V_\alpha^0(j',S,0,0,\ldots,0;n) \\
&\quad \leq C(i) + K(S,0) + \alpha V_\alpha^0(-,S+1,0,0,\ldots,0;n) \\
&\quad = \langle V_\alpha^0(i,S,0,\ldots,0;n+1)\rangle_{2\text{-nd}} .
\end{aligned}$$

Thus, the claim holds for the corresponding second terms for n+1. In a similar fashion, the claim can be proved for other corresponding terms, and also for the case $k = 1$. □

Suppose all the conditions in Theorem 3.7 hold. Then we notice that both Lemmas 3.13 and 3.14 hold. Furthermore, notice that as $A(i) - C(i)$ is nondecreasing in i $(0 \leq i \leq I)$, and since both the second and fourth terms of the right-hand side of $V_\alpha^k(i,s,s_0,\ldots,s_I;n)$ in (3.4) depend on i only through $C(i)$,

$$\min\{\langle V_\alpha^k(i,s,s_0,\ldots,s_I;n)\rangle_{1\text{-st}}, \langle V_\alpha^k(i,s,s_0,\ldots,s_I;n)\rangle_{3\text{-rd}}\}$$
$$- \min\{\langle V_\alpha^k(i,s,s_0,\ldots,s_I;n)\rangle_{2\text{-nd}}, \langle V_\alpha^k(i,s,s_0,\ldots,s_I;n)\rangle_{4\text{-th}}\}$$

is nondecreasing in i $(0 \leq i \leq I)$ by Lemma 3.13. This provides the following: For each fixed $(s, s_0, \ldots, s_I, k)$, there exists a critical number on i, say i^* depending on $(s, s_0, \ldots, s_I, k)$, such that for all states with $i < i^*$, exchanging an operating machine for a suitable spare unit is optimal, and for all states with $i \geq i^*$, repairing an operating machine (or sending it to the repair shop) and introducing a suitable spare unit is optimal. Similarly, if Lemma 3.14 holds, then

$$\min\{\langle V_\alpha^k(i,s,m-s,s_1,\ldots,s_I;n)\rangle_{1\text{-st}},\ \langle V_\alpha^k(i,s,m-s,s_1,\ldots,s_I;n)\rangle_{2\text{-nd}}\}$$

$$-\ \min\{\langle V_\alpha^k(i,s,m-s,s_1,\ldots,s_I;n)\rangle_{3\text{-rd}},\ \langle V_\alpha^k(i,s,m-s,s_1,\ldots,s_I;n)\rangle_{4\text{-th}}\}$$

is nondecreasing in s $(0 \leq s \leq m)$, since both the third and fourth terms depend on s only through $K(s,k)$. This provides the following: For each fixed m $(1 \leq m \leq S)$, $0 \leq i \leq I$, and $(s_1, \ldots, s_I)$ with $\sum_{j=1}^{I} s_j = S-m$, $s_j \geq 0$ $(1 \leq j \leq I)$, there exists a critical number on s, say s^* depending on all m, i, and $(s_1, \ldots, s_I)$, such that for all the states with $s < s^*$, closing the repair service gate is optimal, and for all the states with $s \geq s^*$, opening the repair service gate is optimal. This enables us to divide each i-s diagram with fixed m, k and $(s_1, \ldots, s_I)$ into 4 regions as shown in Fig. 3.5. Moreover, the optimal region for the decision of repairing an operating machine and opening the repair service gate is shown to be rectangle as in the corresponding model without exchange options.

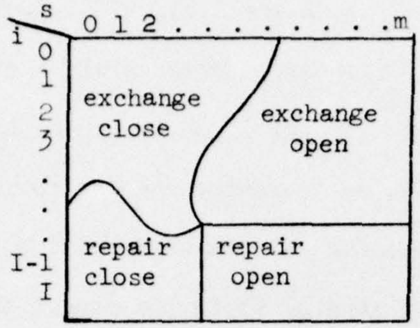

Figure 3.5. A Typical Optimal Policy with Exchange Options.

Thus far, the structure of an optimal policy for this problem resembles that of the simplified maintenance model with control of queue, where a two-dimensional control limit policy is assured to be optimal. But we must realize that the statement "exchanging is optimal" implies only that exchanging an operating machine for a suitable spare unit is optimal, and it does not specify which spare unit should be introduced. Similarly, there remains an ambiguity when it is said that "repairing is optimal." It seems reasonable to consider that a spare unit whose condition is the best among all the available spare units should be introduced to minimize the total cost, but it has yet to be proved. What we can see from the structure of the formulation of the model are the following: In the repair-close optimal region, the condition of a spare machine to be introduced as an operating machine does not depend on the condition of the previously operating machine. In the exchange-open optimal region, the condition of a spare machine to be exchanged with an operating machine, does not depend on the number of machines in the repair system. In the repair-open optimal region, the condition of a spare machine to be introduced depends neither on the number of machines in the repair system nor on the operating condition of the previously operating machine.

3.6. Computational Remarks

All the models treated in this paper are Markov decision models, and the usual techniques to compute an optimal policy for the Markov decision process such as the policy improvement procedure and the LP approach are applicable here too. However, when an optimal policy is known to possess some special simple structure, it seems reasonable to explore whether the computational work will be significantly reduced if the search for an optimal policy can be performed among the policies with the same simple structure. A policy improvement algorithm among stationary control limit policies has been introduced in [5], but the conditions needed for the application of the algorithm are too restrictive to be applied to our models. A policy improvement hybrid algorithm, seen also in [5], would be a suitable method. In this algorithm, better policies are searched iteratively among a set of policies with the desired special structure as long as it is possible, and then usual policy improvement procedure is applied to obtain an optimal policy. Although this algorithm has been investigated to obtain an optimal simple structured control limit policy, it can be easily modified to solve more complicated models such as our models treated here.

3.7. Conclusions

In this paper, two different kinds of discrete time maintenance models with repair shops have been studied. In each model, sufficient conditions have been obtained under which the optimality of some kind of control limit policy is assured.

The first type of model has a multiple number of repair shops in the system. When information on the type of repair required is available at the moment of each decision, simple optimal maintenance policies are obtained under the mild conditions. On the other hand very restrictive conditions are required for the case when such information is not available.

In the next model considered, the action space was expanded to include the option of opening or closing the repair service gate, as well as the option of repairing or leaving an operating machine alone. A two-dimensional control limit policy was defined, and the optimality of such a control limit policy was assured when the repair time of a machine is negligibly small, a very reasonable condition. When the repair time is not negligible, but has a geometric distribution, a restrictive condition on the holding costs must be satisfied.

Since the discrete time queueing control problem has not been fully studied, there are several extensions that can be made on our second control of queue model. Controlling the queue length by changing the repair service rate, controlling a multiple number of repair service stations by opening or closing them, or assuming the repair time distribution is other than geometric are some topics for future research.

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20. Abstract

In this paper, two different kinds of discrete time maintenance models with repair shops are studied.

The first type of model has a multiple number of repair shops in the system. When information on the type of repair required is available at the moment of each decision, simple optimal maintenance policies are obtained under the mild conditions. On the other hand very restrictive conditions are required for the case when such information is not available.

In the next model considered, the action space is expanded to include the option of opening or closing the repair service gate, as well as the option of repairing or leaving an operating machine alone. A two-dimensional control limit policy is defined, and the optimality of such a control limit policy is assured when the repair time of a machine is negligibly small, a very reasonable condition. When the repair time is not negligible, but has a geometric distribution, a restrictive condition on the holding costs must be satisfied.

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